

STABLE CATEGORIES OF GRADED MAXIMAL COHEN-MACAULAY MODULES OVER NONCOMMUTATIVE QUOTIENT SINGULARITIES

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ABSTRACT. Tilting objects play a key role in the study of triangulated categories. A famous result due to Iyama and Takahashi asserts that the stable categories of graded maximal Cohen-Macaulay modules over quotient singularities have tilting objects. This paper proves a noncommutative generalization of Iyama and Takahashi's theorem using noncommutative algebraic geometry. Namely, if S is a noetherian AS-regular Koszul algebra and G is a finite group acting on S such that S^G is a "Gorenstein isolated singularity", then the stable category $\underline{\text{CM}}^{\mathbb{Z}}(S^G)$ of graded maximal Cohen-Macaulay modules has a tilting object. In particular, the category $\underline{\text{CM}}^{\mathbb{Z}}(S^G)$ is triangle equivalent to the derived category of a finite dimensional algebra.

1. INTRODUCTION

Triangulated categories are increasingly important in many areas of mathematics including algebraic geometry and representation theory. There are two major classes of triangulated categories, namely, derived categories of abelian categories and stable categories of Frobenius categories. For example, derived categories of coherent sheaves have been extensively studied in algebraic geometry, and stable module categories of selfinjective algebras have been extensively studied in representation theory of finite dimensional algebras.

In the study of triangulated categories, tilting objects play a key role. They often enable us to realize abstract triangulated categories as concrete derived categories of modules over algebras. One of the remarkable results on the existence of tilting objects has been obtained by Iyama and Takahashi.

Theorem 1.1. [6, Theorem 2.7, Corollary 2.10] *Let $S = k[x_1, \dots, x_d]$ be a polynomial algebra over an algebraically closed field k of characteristic 0 such that $\deg x_i = 1$ and $d \geq 2$. Let G be a finite subgroup of $\text{SL}(d, k)$ acting linearly on S , and S^G the fixed subalgebra of S . Assume that S^G is an isolated singularity. If we define the graded S^G -module*

$$T := \bigoplus_{i=1}^d f_* \Omega_S^i(i)$$

where $f : S^G \rightarrow S$ is the inclusion, then the stable category $\underline{\text{CM}}^{\mathbb{Z}}(S^G)$ of graded maximal Cohen-Macaulay modules has a tilting object $[T]_{\text{CM}}$, where $[T]_{\text{CM}}$ is the maximal direct summand of T which is a graded maximal Cohen-Macaulay module. As a consequence, there exists a finite dimensional algebra Γ of finite global dimension such that

$$\underline{\text{CM}}^{\mathbb{Z}}(S^G) \cong \mathcal{D}^b(\text{mod } \Gamma).$$

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The stable categories of graded maximal Cohen-Macaulay modules are crucial objects studied in representation theory of algebras (see [6], [2] etc.) and also attract attention from the viewpoint of Kontsevich's homological mirror symmetry conjecture (see [8], [9] etc.). The aim of the present paper is to generalize Theorem 1.1 to the noncommutative case using noncommutative algebraic geometry.

For the rest, we basically follow the terminologies and notations in [16, Section 1.1] (see subsection 1.1 below). Throughout this paper, k denotes an algebraically closed field. Let A be a graded algebra over k , and G a finite subgroup of $\text{GrAut } A$ such that $\text{char } k$ does not divide $|G|$. In this case, kG is a semisimple algebra. Two idempotent elements

$$e := \frac{1}{|G|} \sum_{g \in G} g, \quad \text{and} \quad e' := 1 - e$$

of kG play crucial roles in this paper. Since $kG \subset A * G$, we often view e, e' as idempotent elements of $A * G$. Moreover, since $e(A * G)e \cong A^G$ as graded algebras, we usually identify $e(A * G)e$ with A^G .

In [7], Jørgensen and Zhang introduced the notion of homological determinant for a graded algebra automorphism g of a noetherian AS-regular algebra S over k , and proved that if G is a finite subgroup of the homological special linear group

$$\text{HSL}(S) = \{g \in \text{GrAut } S \mid \text{the homological determinant of } g \text{ is } 1\}$$

on S , then S^G is a noncommutative Gorenstein algebra (called an AS-Gorenstein algebra).

In [21], the second author introduced a notion of graded isolated singularity for noncommutative graded algebras, which agrees with the usual notion of isolated singularity if the algebra is commutative and generated in degree 1, and found some nice properties of such algebras. For a noetherian AS-regular algebra S over k and a finite subgroup $G \leq \text{GrAut } S$, it was shown in [16] that the condition that $S * G/(e)$ is finite dimensional over k is closely related to the noncommutative graded isolated singularity property of S^G . More precisely, for $G \leq \text{HSL}(S)$, it was proved that $S * G/(e)$ is finite dimensional over k if and only if S^G is a noncommutative graded isolated singularity and $S * G \cong \underline{\text{End}}_{S^G}(S)$ (see [16, Theorem 3.10]).

If $S = k[x_1, \dots, x_d]$ is the polynomial algebra and G is a finite group acting on S , then $\text{Spec } S^G \cong \mathbb{A}^d/G$ is a quotient singularity. Further, if $\deg x_i = 1$ for all i , then $\text{tails } S \cong \text{coh } \mathbb{P}^{d-1}$ the category of coherent sheaves on \mathbb{P}^{d-1} , and it is well-known that $\mathcal{O}_{\mathbb{P}^{d-1}}, \mathcal{O}_{\mathbb{P}^{d-1}}(1), \dots, \mathcal{O}_{\mathbb{P}^{d-1}}(d-1)$ is a full strong exceptional sequence for $\mathcal{D}^b(\text{coh } \mathbb{P}^{d-1})$ so that $\bigoplus_{i=0}^{d-1} \mathcal{O}_{\mathbb{P}^{d-1}}(i)$ is a tilting object for $\mathcal{D}^b(\text{coh } \mathbb{P}^{d-1})$. Suppose that a finite group G acts on a noetherian AS-regular algebra S over k of Gorenstein parameter ℓ . Since S is commutative if and only if S is a polynomial algebra, the fixed subalgebra S^G is regarded as a noncommutative quotient singularity and the noncommutative projective scheme $X := \text{Proj}_{nc} S$ associated to S is regarded as a quantum projective space. The inclusion map $f : S^G \rightarrow S$ induces a functor $f_* : \text{tails } S \rightarrow \text{tails } S^G$. If G is non-trivial, then $f_* \mathcal{O}_X, f_* \mathcal{O}_X(1), \dots, f_* \mathcal{O}_X(\ell-1)$ is no longer an exceptional sequence for $\mathcal{D}^b(\text{tails } S^G)$, however, the following result shows that $\bigoplus_{i=0}^{\ell-1} f_* \mathcal{O}_X(i)$ is a tilting object for $\mathcal{D}^b(\text{tails } S^G)$ if S^G is an “isolated singularity”.

Theorem 1.2. [16, Theorem 3.14] *Let S be a noetherian AS-regular algebra over k of dimension $d \geq 2$ and of Gorenstein parameter ℓ , and G a finite subgroup of $\text{GrAut } S$ such that $\text{char } k$ does not divide $|G|$. If $S * G/(e)$ is finite dimensional over k , then*

$$\bigoplus_{i=0}^{\ell-1} f_* \mathcal{O}_X(i)$$

is a tilting object in $\mathcal{D}^b(\text{tails } S^G)$ where $X := \text{Proj}_{nc} S$ and $f : S^G \rightarrow S$ is the inclusion.

There exists another full strong exceptional sequence $\Omega_{\mathbb{P}^{d-1}}^{d-1}(d-1), \dots, \Omega_{\mathbb{P}^{d-1}}^1(1), \Omega_{\mathbb{P}^{d-1}}^0 = \mathcal{O}_{\mathbb{P}^{d-1}}$ for $\mathcal{D}^b(\text{coh } \mathbb{P}^{d-1})$ so that $\bigoplus_{i=0}^{d-1} \Omega_{\mathbb{P}^{d-1}}^i(i)$ is a tilting object. In the setting of the above theorem, if G is non-trivial, then $f_*\Omega_X^{d-1}(d-1), \dots, f_*\Omega_X^1(1), f_*\Omega_X^0$ is no longer an exceptional sequence for $\mathcal{D}^b(\text{tails } S^G)$, however, we will show in this paper that $\bigoplus_{i=0}^{d-1} f_*\Omega_X^i(i)$ is a tilting object for $\mathcal{D}^b(\text{tails } S^G)$ if S is Koszul.

Theorem 1.3. (Theorem 3.20) *Let S be a noetherian AS-regular Koszul algebra of dimension d over k , and G a finite subgroup of $\text{GrAut } S$ such that $\text{char } k$ does not divide $|G|$. If $S * G/(e)$ is finite dimensional over k , then*

$$\bigoplus_{i=0}^{d-1} f_*\Omega_X^i(i)$$

is a tilting object in $\mathcal{D}^b(\text{tails } S^G)$ where $X := \text{Proj}_{nc} S$ and $f : S^G \rightarrow S$ is the inclusion. As a consequence, there exists a finite dimensional algebra Λ of finite global dimension such that

$$\mathcal{D}^b(\text{tails } S^G) \cong \mathcal{D}^b(\text{mod } \Lambda).$$

In the setting of the above theorem, the skew group algebra $S * G$ is an AS-regular Koszul algebra of dimension d over kG (Proposition 2.14 and Lemma 2.21), and the Koszul dual algebra $(S * G)^\dagger$ is a Frobenius Koszul algebra of Gorenstein parameter $-d$ (Proposition 2.27). Thus one can define the stable category $\underline{\text{grmod}}(S * G)^\dagger$. Then we have an equivalence $\overline{K} : \underline{\text{grmod}}(S * G)^\dagger \rightarrow \mathcal{D}^b(\text{tails } S * G)$ called the BGG correspondence (Proposition 3.3). If $S * G/(e)$ is finite dimensional over k , then we also have the equivalence $(-)e : \mathcal{D}^b(\text{tails } S * G) \rightarrow \mathcal{D}^b(\text{tails } S^G)$ (Proposition 3.18). The key point of our proof is to show that under the equivalences

$$\underline{\text{grmod}}(S * G)^\dagger \xrightarrow[\overline{K}]{\sim} \mathcal{D}^b(\text{tails } S * G) \xrightarrow[\text{(-)}e]{\sim} \mathcal{D}^b(\text{tails } S^G),$$

the tilting object in $\underline{\text{grmod}}(S * G)^\dagger$ which was obtained by Yamaura [22] corresponds to the object $\bigoplus_{i=0}^{d-1} f_*\Omega_X^i(i)$ in $\mathcal{D}^b(\text{tails } S^G)$ (Lemma 3.11, Corollary 3.15 and Theorem 3.20).

We define a graded kG - $S * G$ bimodule U by

$$U := \bigoplus_{i=1}^d \Omega_{S * G}^i kG(i).$$

Using Theorem 1.3, we will show the existence of a tilting object of the stable category $\underline{\text{CM}}^\mathbb{Z}(S^G)$ if S^G is a ‘‘Gorenstein isolated singularity’’. The main result of this paper is as follows.

Theorem 1.4. (Theorem 4.10, Theorem 4.17) *Let S be a noetherian AS-regular Koszul algebra of dimension $d \geq 2$ over k , and G a finite subgroup of $\text{HSL}(S)$ such that $\text{char } k$ does not divide $|G|$. If $S * G/(e)$ is finite dimensional over k , then*

$$e'Ue$$

is a tilting object in $\underline{\text{CM}}^\mathbb{Z}(S^G)$. As a consequence, there exists a finite dimensional algebra $\Gamma = e'\Lambda e'$ of finite global dimension such that

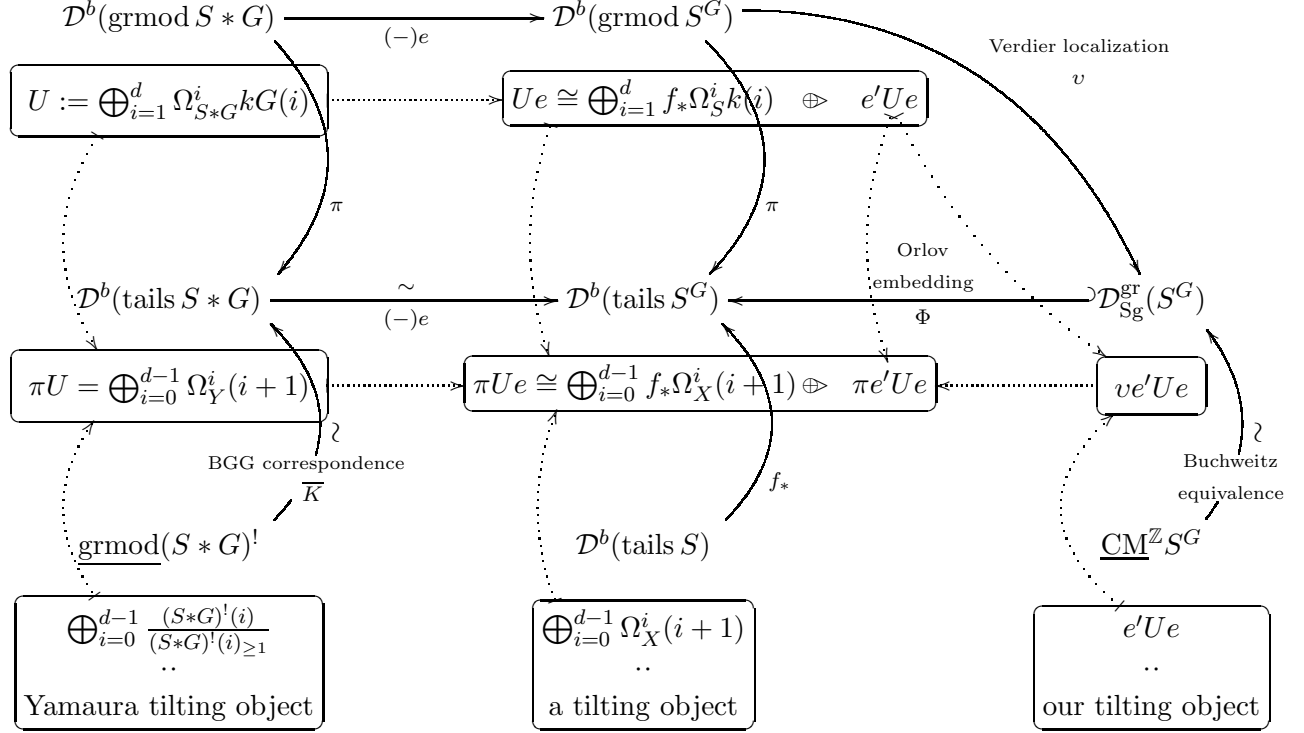
$$\underline{\text{CM}}^\mathbb{Z}(S^G) \cong \mathcal{D}^b(\text{mod } \Gamma).$$

If S is a commutative AS-regular Koszul algebra of dimension d over k , then $S = k[x_1, \dots, x_d]$ with $\deg x_i = 1$ for all i . In this case,

- $\text{HSL}(S)$ coincides with $\text{SL}(d, k)$.
- $S * G/(e)$ is finite dimensional over k if and only if S^G is a (graded) isolated singularity (see [16, Corollary 3.11]).

- $e'Ue = [T]_{\text{CM}}$ (see [6, Proof of Theorem 2.9]).

Thus it follows that our result is a generalization of Theorem 1.1. However, our proof is different from the original one given in [6]. Thanks to Theorem 1.3, we can give a more conceptual proof in terms of triangulated categories. The following is a flow diagram of objects which are essential for this paper.



where $X := \text{Proj}_{nc} S$, $Y := \text{Proj}_{nc} S * G$, and $M \oplus N$ means that N is a direct summand of M .

In the last section of this paper, we will give an illustrative example.

1.1. Terminologies and Notations. We first introduce some basic terminologies and notations used in this paper. Let k be an algebraically closed field. Unless otherwise stated, a graded algebra means an \mathbb{N} -graded algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ over k . The group of graded k -algebra automorphisms of A is denoted by $\text{GrAut } A$. We denote by $\text{GrMod } A$ the category of graded right A -modules, and by $\text{grmod } A$ the full subcategory consisting of finitely presented modules. Note that if A is graded right coherent, then $\text{grmod } A$ is an abelian category. Morphisms in $\text{GrMod } A$ are right A -module homomorphisms of degree zero. Graded left A -modules are identified with graded A^o -modules where A^o is the opposite graded algebra of A . For $M \in \text{GrMod } A$ and $n \in \mathbb{Z}$, we define $M_{\geq n} = \bigoplus_{i \geq n} M_i \in \text{GrMod } A$, and $M(n) \in \text{GrMod } A$ by $M(n) = M$ as an ungraded right A -module with the new grading $M(n)_i = M_{n+i}$. The rule $M \mapsto M(n)$ is a k -linear autoequivalence for $\text{GrMod } A$ and $\text{grmod } A$, called the shift functor. For $M, N \in \text{GrMod } A$, we write the vector space $\text{Ext}_A^i(M, N) := \text{Ext}_{\text{GrMod } A}^i(M, N)$ and the graded vector space

$$\underline{\text{Ext}}_A^i(M, N) := \bigoplus_{n \in \mathbb{Z}} \text{Ext}_A^i(M, N(n)).$$

We denote by D the k -vector space dual. For a graded right (resp. left) A -module M , we also denote by $DM := \underline{\text{Hom}}_k(M, k)$ the graded k -vector space dual of M by abuse of notation. Note

that DM has a graded left (resp. right) A -module structure. We say that $M \in \text{GrMod } A$ is torsion if, for any $m \in M$, there exists $n \in \mathbb{N}$ such that $mA_{\geq n} = 0$. We denote by $\text{Tors } A$ the full subcategory of $\text{GrMod } A$ consisting of torsion modules. We write $\text{Tails } A$ for the quotient category $\text{GrMod } A / \text{Tors } A$. The quotient functor is denoted by $\pi : \text{GrMod } A \rightarrow \text{Tails } A$. The objects in $\text{Tails } A$ will be denoted by script letters, like $\mathcal{M} = \pi M$. If A is graded right coherent, then we define $\text{tails } A := \text{grmod } A / \text{tors } A$ where $\text{tors } A = \text{Tors } A \cap \text{grmod } A$ is the full subcategory of $\text{grmod } A$ consisting of finite dimensional modules. We call $\text{tails } A$ the noncommutative projective scheme associated to A since $\text{tails } A \cong \text{coh}(\text{Proj } A)$ the category of coherent sheaves on $\text{Proj } A$ if A is commutative and generated in degree 1. The shift functor on $\text{GrMod } A$ induces an autoequivalence $(n) : \mathcal{M} \mapsto \mathcal{M}(n)$ for $\text{Tails } A$ and $\text{tails } A$, also call the shift functor. For $\mathcal{M}, \mathcal{N} \in \text{Tails } A$, we write the vector space $\text{Ext}_{\mathcal{A}}^i(\mathcal{M}, \mathcal{N}) := \text{Ext}_{\text{Tails } A}^i(\mathcal{M}, \mathcal{N})$ and the graded vector space

$$\underline{\text{Ext}}_{\mathcal{A}}^i(\mathcal{M}, \mathcal{N}) := \bigoplus_{n \in \mathbb{Z}} \text{Ext}_{\mathcal{A}}^i(\mathcal{M}, \mathcal{N}(n)).$$

We remark that, in many papers (eg. [6]), $\underline{\text{Hom}}$ and $\underline{\text{End}}$ denote Hom and End in the stable categories, however, in this paper, $\underline{\text{Hom}}$ and $\underline{\text{End}}$ always mean graded Hom and graded End as defined above, following the tradition of noncommutative algebraic geometry starting from [3].

2. ALGEBRAS TO STUDY

In this section, we will give definitions and basic properties of algebras to study in this paper. Some of the results in this section and the next section are essentially not new but rather slight generalizations of results in [11] and [12] etc. We will give proofs for some of such results to make this paper as self-contained as possible for the reader who is not an expert.

2.1. Koszul Algebras.

Definition 2.1. Let A be a graded algebra. A linear resolution of $M \in \text{GrMod } A$ is a graded projective resolution of M

$$\cdots \rightarrow P^2 \rightarrow P^1 \rightarrow P^0 \rightarrow M \rightarrow 0$$

where each P^i is a graded projective right A -module generated in degree i .

A graded algebra A is called Koszul if

- (1) A is locally finite, that is, $\dim_k A_i < \infty$ for all $i \in \mathbb{N}$,
- (2) A_0 is a semisimple algebra, and
- (3) $A_0 := A/A_{\geq 1} \in \text{GrMod } A$ has a linear resolution.

In many literature, the definition of a Koszul algebra A requires for A_0 to be a finite direct product of k . On the other hand, if A is a Koszul algebra defined as above, then A_0 is a finite dimensional semisimple algebra over an algebraically closed field k , so A_0 is isomorphic to a finite direct product of full matrix algebras over k . Thus a Koszul algebra in this paper is more general than that in many literature, but more restrictive than that in [4]. In particular, A_0 is a symmetric algebra, that is, $D(A_0) \cong A_0$ as A_0 - A_0 bimodules, so we have the following result for our Koszul algebras.

Lemma 2.2. *Let A be a Koszul algebra.*

- (1) *For every finite dimensional A_0 - A_0 bimodule M , $\text{Hom}_{A_0}(M, A_0) \cong \text{Hom}_{A_0^0}(M, A_0) \cong D(M)$ as A_0 - A_0 bimodules.*
- (2) *For every locally finite A - A_0 bimodule M , $\underline{\text{Hom}}_{A_0}(M, A_0) \cong D(M)$ as graded A_0 - A bimodules.*
- (3) *For every locally finite A_0 - A bimodule M , $\underline{\text{Hom}}_{A_0^0}(M, A_0) \cong D(M)$ as graded A - A_0 bimodules.*

Due to the above lemma, we identify the right dual and the left dual, which were carefully distinguished in [4] (see [4, Section 2.7]). Another property of our Koszul algebra is as follows. A graded algebra A is called right (resp. left) finite if A_i are finitely generated as right (resp. left) A_0 -modules for all $i \in \mathbb{N}$. It is easy to see that a graded algebra A is locally finite if and only if A is right (or left) finite and $\dim_k A_0 < \infty$, so our Koszul algebra is both left and right finite. Our Koszul algebra enjoys the following usual properties of a Koszul algebra.

Lemma 2.3. [4, Proposition 2.2.1] *Let A be a graded algebra. Then A is Koszul if and only if A° is Koszul.*

Definition 2.4. For a graded algebra A , we define the graded algebra

$$A^! := E_A(A_0) := \bigoplus_{i \in \mathbb{N}} \underline{\text{Ext}}_A^i(A_0, A_0),$$

and the functor $E_A : \text{GrMod } A \rightarrow \text{GrMod}(A^!)^\circ$ by

$$E_A(M) := \bigoplus_{i \in \mathbb{N}} \underline{\text{Ext}}_A^i(M, A_0).$$

Lemma 2.5. [4, Proposition 2.9.1, Theorem 2.10.2] *If A is a Koszul algebra, then*

- (1) $A^!$ is also a Koszul algebra, called the Koszul dual of A , and
- (2) $(A^!)^! \cong A$ as graded algebras.

If A is a Koszul algebra, then it is a quadratic algebra by [4, Corollary 2.3.3], so we may write $A = T_{A_0}(A_1)/(R)$ for some $R \subset A_1 \otimes_{A_0} A_1$.

Lemma 2.6. [4, Theorem 2.6.1] *If $A = T_{A_0}(A_1)/(R)$ is a Koszul algebra where $R \subset A_1 \otimes_{A_0} A_1$, then the linear resolution of A_0 over A is given by $(K_i^i \otimes_{A_0} A, d^i)$ where*

$$K_i^i := \bigcap_{s+t+2=i} A_1^{\otimes_{A_0} s} \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes_{A_0} t} \subset A_1^{\otimes_{A_0} i}$$

and d^i is the restriction of the map

$$A_1^{\otimes_{A_0} i} \otimes_{A_0} A \rightarrow A_1^{\otimes_{A_0} i-1} \otimes_{A_0} A; v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i \otimes a \mapsto v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i a.$$

The full subcategory of $\text{GrMod } A$ consisting of modules having linear resolutions P^\bullet such that each P^i is a finitely generated graded projective right A -module (generated in degree i) is denoted by $\text{lin } A$.

Lemma 2.7. *If A is a Koszul algebra, then $A_0 \in \text{lin } A$ and $A_0 \in \text{lin } A^\circ$.*

Proof. By Lemma 2.6, the linear resolution K^\bullet of $A_0 \in \text{GrMod } A$ is of the form $K^i = K_i^i \otimes_{A_0} A$ where each K_i^i is an A_0 subbimodule of $A_1^{\otimes_{A_0} i}$. Since $\dim_k A_1 < \infty$, $\dim_k K_i^i < \infty$. It follows that K_i^i is finitely generated as a right A_0 -module, so K^i is finitely generated as a graded right A -module, hence $A_0 \in \text{lin } A$. By Lemma 2.3, $A_0 \in \text{lin } A^\circ$. \square

Definition 2.8. Let A be a graded algebra and $M \in \text{GrMod } A$. A graded projective resolution (P^\bullet, d) is called minimal if $d^i \otimes_A A_0 = 0$ for every $i \in \mathbb{N}^+$.

If A is a graded algebra such that A_0 is semisimple, then it is known that a minimal projective resolution of $M \in \text{GrMod } A$ is unique up to isomorphism. Clearly, every linear resolution is a minimal resolution, so we have the following result.

Lemma 2.9. *Let A be a Koszul algebra. A linear resolution of $M \in \text{GrMod } A$ is unique up to isomorphism if it exists.*

Definition 2.10. Let A be a Koszul algebra and (P^\bullet, d) the linear resolution of $M \in \text{lin } A$. The i -th syzygy of M is defined by $\Omega^i M := \text{Im } d^i$ for $i \in \mathbb{N}^+$.

Remark 2.11. Let A be a Koszul algebra. Since the linear resolution (K^\bullet, d) of A_0 is a complex of graded A_0 - A bimodules (see [4, Remark (1) to Definition 1.2.1]), $\Omega^i A_0$ has a graded A_0 - A bimodule structure.

2.2. Skew Group Algebras.

Definition 2.12. Let R be an algebra and $G \leq \text{Aut } R$ a subgroup. We define the skew group algebra of R by G by $R * G = R \otimes_k kG$ as a k -vector space with the multiplication

$$(a \otimes g) \cdot (b \otimes h) = ag(b) \otimes gh.$$

An element $a \otimes g \in R * G$ is usually denoted by $a * g$.

Similarly, we define $G * R = kG \otimes_k R$ as a k -vector space with the multiplication

$$(g \otimes a) \cdot (h \otimes b) = gh \otimes h^{-1}(a)b.$$

The proof of the following lemma is left to the reader.

Lemma 2.13. *Let R be an algebra and $G \leq \text{Aut } R$ a subgroup. Then the following hold:*

- (1) $(R * G)^o \cong R^o * G$ as algebras.
- (2) $G * R \cong R * G^o$ as algebras.

Let A be a graded algebra, and $G \leq \text{GrAut } A$ a finite subgroup. If $\text{char } k$ does not divide $|G|$, then we define $e := \frac{1}{|G|} \sum_{g \in G} g \in kG \subset A * G$. By [16], A has a graded right $A * G$ -module structure by $c \cdot (a * g) := g^{-1}(ca)$, and $A \cong e(A * G)$ as graded right $A * G$ -modules. Since $A_{\geq i}$ is a submodule of A as a graded right $A * G$ -module, $A_0 := A/A_{\geq 1} \cong e(A * G)_0 \cong e(A_0 * G)$ has a graded right $A * G$ -module structure. We also define a graded left $A * G$ -module structure on A by $(a * g) \cdot c := ag(c)$. Since $A_{\geq i}$ is a submodule of A as a graded left $A * G$ -module, $A_0 := A/A_{\geq 1} \cong (A * G)_0 e \cong (A_0 * G)e$ has a graded left $A * G$ -module structure as well.

If M is a graded left $A * G$ -module and V is a left kG -module, then we define the graded left $A * G$ -module structure on $M \otimes_k V$ by $(a * g) \cdot (m \otimes v) := (a * g)m \otimes gv$ (cf. [11]). Since A has a graded left $A * G$ -module structure, $A \otimes_k kG$ has a graded left $A * G$ -module structure by $(a * g)(b \otimes h) = (a * g)b \otimes gh = ag(b) \otimes gh$. It follows that $A \otimes_k kG \cong A * G$ and $A_0 \otimes_k kG \cong (A * G)_0$ as graded left $A * G$ -modules. We can define the action of G^o on $A^!$ as explained in [19]. We remark that this action is the “inverse” of the action in [11].

The next proposition was essentially proved in [11], however, the definition of a Koszul algebra given in [11] is slightly different from ours, so we will provide our own proof.

Proposition 2.14. (cf. [11, Theorem 10, Theorem 14]) *Let A be a graded algebra and $G \leq \text{GrAut } A$ a finite subgroup such that $\text{char } k$ does not divide $|G|$. If A_0 has a graded projective resolution consisting of finitely generated graded projective right $A * G$ -modules, then the following hold:*

- (1) $(A * G)^! \cong G * A^!$ as graded algebras.
- (2) A is Koszul if and only if $A * G$ is Koszul.

Proof. (1) If A_0 has a graded projective resolution consisting of finitely generated graded projective left $A * G$ -modules, then we have an isomorphism

$$\begin{aligned} \underline{\text{Ext}}_{A^o}^i(A_0, A_0) \otimes_k kG &\cong \underline{\text{Ext}}_{(A * G)^o}^i(A_0 \otimes_k kG, A_0 \otimes_k kG) \\ &\cong \underline{\text{Ext}}_{(A * G)^o}^i((A * G)_0, (A * G)_0) \end{aligned}$$

of graded vector spaces for each $i \in \mathbb{N}$ by [11, Lemma 8, Proposition 9] (one can check that the isomorphism θ in [11, Lemma 8] is given by

$$\theta(f \otimes w)(p \otimes g) = (g^{-1} \star f)(p) \otimes gw^{-1}$$

in our setting where \star means the group action, and can also check that it preserves the grading). Furthermore we have an isomorphism

$$E_{A^\circ}(A_0)^\circ * G^\circ \cong E_{(A * G)^\circ}((A * G)_0)^\circ$$

as graded algebra proved by the same arguments as in [11, Lemma 10]. Now we replace A by A° . Since $(A^\circ * G)^\circ \cong A * G$ by Lemma 2.13, if A_0 has a graded projective resolution consisting of finitely generated graded projective right $A * G$ -modules, then we have an isomorphism

$$\underline{\text{Ext}}_A^i(A_0, A_0) \otimes_k kG \cong \underline{\text{Ext}}_{(A^\circ * G)^\circ}^i((A * G)_0, (A * G)_0) \cong \underline{\text{Ext}}_{A * G}^i((A * G)_0, (A * G)_0)$$

of graded vector spaces for each $i \in \mathbb{N}$, and an isomorphism

$$(A * G)^\dagger = E_{A * G}((A * G)_0) \cong E_{(A^\circ * G)^\circ}((A * G)_0) \cong (E_A(A_0)^\circ * G^\circ)^\circ \cong E_A(A_0) * G^\circ = G * A^\dagger$$

as graded algebras by Lemma 2.13.

(2) Since $(A * G)_i \cong A_i \otimes_k kG$ as vector spaces, A is locally finite if and only if $A * G$ is locally finite. Since $\text{char } k$ does not divide $|G|$, $\text{gldim}(A * G)_0 = \text{gldim } A_0 * G = \text{gldim } A_0$, so A_0 is semisimple if and only if $(A * G)_0$ is semisimple.

Since

$$\begin{aligned} \text{Ext}_{A * G}^i((A * G)_0, (A * G)_0(j)) &\cong \underline{\text{Ext}}_{A * G}^i((A * G)_0, (A * G)_0)_j \\ &\cong (\underline{\text{Ext}}_A^i(A_0, A_0) \otimes_k kG)_j \\ &\cong \underline{\text{Ext}}_A^i(A_0, A_0)_j \otimes_k kG \\ &\cong \text{Ext}_A^i(A_0, A_0(j)) \otimes_k kG, \end{aligned}$$

$(A * G)_0 \in \text{lin } A * G$ if and only if $\text{Ext}_{A * G}^i((A * G)_0, (A * G)_0(j)) = 0$ unless $i + j = 0$ if and only if $\text{Ext}_A^i(A_0, A_0(j)) = 0$ unless $i + j = 0$ if and only if $A_0 \in \text{lin } A$ by [4, Proposition 2.14.2]. \square

Let A be a graded algebra, $G \leq \text{GrAut } A$ a finite subgroup such that $\text{char } k$ does not divide $|G|$, and $e := \frac{1}{|G|} \sum_{g \in G} g \in kG$ an idempotent. We define a graded right A -module structure on $(A * G)e$ by $(a * g)e \cdot b = (ab * g)e$. Since

$$((a * g)e \cdot b) \cdot c = (ab * g)e \cdot c = (abc * g)e = (a * g)e \cdot bc,$$

it is a well-defined graded right A -module structure.

Lemma 2.15. *If A is a graded algebra, $G \leq \text{GrAut } A$ is a finite subgroup such that $\text{char } k$ does not divide $|G|$, and $e := \frac{1}{|G|} \sum_{g \in G} g \in kG$, then the map $\phi : (A * G)e \rightarrow A$ defined by $\phi((a * g)e) = a$ is an isomorphism in $\text{GrMod } A$.*

Proof. Define a map $\psi : A \rightarrow (A * G)e$ by $\psi(a) = (a * e)e$ where $e \in G$ is the identity. Since

$$\psi(ab) = (ab * e)e = (a * e)e \cdot b = \psi(a) \cdot b,$$

ψ is a graded right A -module homomorphism. By [16, Lemma 1.4], ψ is bijective, so $\psi : A \rightarrow (A * G)e$ is an isomorphism of graded right A -modules. Since $\phi(\psi(a)) = \phi((a * e)e) = a$ for every $a \in A$, $\phi = \psi^{-1} : (A * G)e \rightarrow A$ is an isomorphism of graded right A -modules. \square

Lemma 2.16. *If A is a graded algebra and $G \leq \text{GrAut } A$, then the map $\varphi : A_1^{\otimes k^i} \otimes_k A * G \rightarrow (A * G)_1^{\otimes k^i} \otimes_{kG} A * G$ defined by*

$$\varphi(v_1 \otimes \cdots \otimes v_i \otimes (a * g)) = (v_1 * e) \otimes \cdots \otimes (v_i * e) \otimes (a * g)$$

is an isomorphism in $\text{GrMod } A * G$ where $\epsilon \in G$ is the identity.

Proof. Define a map $\psi : (A * G)_1^{\otimes_{kG} i} \otimes_{kG} A * G \rightarrow A_1^{\otimes_k i} \otimes_k A * G$ by

$$\psi((v_1 * g_1) \otimes (v_2 * g_2) \otimes \cdots \otimes (v_i * g_i) \otimes (a * g)) = v_1 \otimes g_1(v_2) \otimes \cdots \otimes g_1 \cdots g_{i-1}(v_i) \otimes (g_1 \cdots g_i(a) * g_1 \cdots g_i g).$$

For $(v_1 * g_1) \otimes (v_2 * g_2) \otimes (a * g) \in (A * G)_1 \otimes_{kG} (A * G)_1 \otimes_{kG} A * G$ and $h \in G$,

$$\begin{aligned} \psi((v_1 * g_1) \cdot h \otimes (v_2 * g_2) \otimes (a * g)) &= \psi((v_1 * g_1)(1 * h) \otimes (v_2 * g_2) \otimes (a * g)) \\ &= \psi((v_1 * g_1 h) \otimes (v_2 * g_2) \otimes (a * g)) \\ &= v_1 \otimes g_1 h(v_2) \otimes (g_1 h g_2(a) * g_1 h g_2 g) \\ &= \psi((v_1 * g_1) \otimes (h(v_2) * h g_2) \otimes (a * g)) \\ &= \psi((v_1 * g_1) \otimes (1 * h)(v_2 * g_2) \otimes (a * g)) \\ &= \psi((v_1 * g_1) \otimes h \cdot (v_2 * g_2) \otimes (a * g)), \end{aligned}$$

so we can see that ψ is well-defined. Moreover, since

$$\begin{aligned} (v_1 * g_1) \otimes (v_2 * g_2) \otimes (a * g) &= (v_1 * \epsilon)(1 * g_1) \otimes (v_2 * g_2) \otimes (a * g) \\ &= (v_1 * \epsilon) \cdot g_1 \otimes (v_2 * g_2) \otimes (a * g) \\ &= (v_1 * \epsilon) \otimes g_1 \cdot (v_2 * g_2) \otimes (a * g) \\ &= (v_1 * \epsilon) \otimes (1 * g_1)(v_2 * g_2) \otimes (a * g) \\ &= (v_1 * \epsilon) \otimes (g_1(v_2) * g_1 g_2) \otimes (a * g) \\ &= (v_1 * \epsilon) \otimes (g_1(v_2) * \epsilon)(1 * g_1 g_2) \otimes (a * g) \\ &= (v_1 * \epsilon) \otimes (g_1(v_2) * \epsilon) \cdot g_1 g_2 \otimes (a * g) \\ &= (v_1 * \epsilon) \otimes (g_1(v_2) * \epsilon) \otimes g_1 g_2 \cdot (a * g) \\ &= (v_1 * \epsilon) \otimes (g_1(v_2) * \epsilon) \otimes (1 * g_1 g_2)(a * g) \\ &= (v_1 * \epsilon) \otimes (g_1(v_2) * \epsilon) \otimes (g_1 g_2(a) * g_1 g_2 g), \end{aligned}$$

we can show that

$$\begin{aligned} &\varphi(\psi((v_1 * g_1) \otimes (v_2 * g_2) \otimes \cdots \otimes (v_i * g_i) \otimes (a * g))) \\ &= \varphi(v_1 \otimes g_1(v_2) \otimes \cdots \otimes g_1 \cdots g_{i-1}(v_i) \otimes (g_1 \cdots g_i(a) * g_1 \cdots g_i g)) \\ &= (v_1 * \epsilon) \otimes (g_1(v_2) * \epsilon) \otimes \cdots \otimes (g_1 \cdots g_{i-1}(v_i) * \epsilon) \otimes (g_1 \cdots g_i(a) * g_1 \cdots g_i g) \\ &= (v_1 * g_1) \otimes (v_2 * g_2) \otimes \cdots \otimes (v_i * g_i) \otimes (a * g). \end{aligned}$$

On the other hand, for $v_1 \otimes v_2 \otimes \cdots \otimes v_i \otimes (a * g) \in A_1^{\otimes_k i} \otimes_k A * G$,

$$\begin{aligned} \psi(\varphi(v_1 \otimes v_2 \otimes \cdots \otimes v_i \otimes (a * g))) &= \psi((v_1 * \epsilon) \otimes (v_2 * \epsilon) \otimes \cdots \otimes (v_i * \epsilon) \otimes (a * g)) \\ &= v_1 \otimes v_2 \otimes \cdots \otimes v_i \otimes (a * g), \end{aligned}$$

so ψ is the inverse of φ , hence φ is a bijection. Clearly, φ is a graded right $A * G$ -module homomorphism, so the result follows. \square

Let A be a graded algebra, $G \leq \text{GrAut } A$ a finite subgroup such that $\text{char } k$ does not divide $|G|$, and $e := \frac{1}{|G|} \sum_{g \in G} g \in kG$. Since $(A * G)e$ is projective as a graded left $A * G$ -module, and $e(A * G)e \cong A^G$ as graded algebras, we have an exact functor $(-)_e := - \otimes_{A * G} (A * G)e : \text{GrMod } A * G \rightarrow \text{GrMod } A^G$. On the other hand, the inclusion map $f : A^G \rightarrow A$ induces an exact functor $f_* : \text{GrMod } A \rightarrow \text{GrMod } A^G$. In fact, a complex C^\bullet of graded right A -modules is exact if and only if the complex $f_* C^\bullet$ of graded right A^G -modules is exact.

Theorem 2.17. *Let A be a connected graded Koszul algebra (that is, $A_0 = k$), $G \leq \text{GrAut } A$ a finite subgroup such that $\text{char } k$ does not divide $|G|$, $e := \frac{1}{|G|} \sum_{g \in G} g \in kG$, and $f : A^G \rightarrow A$ the inclusion map. If $A * G$ is a Koszul algebra, then $(\Omega_{A * G}^i kG)e \cong f_* \Omega_A^i k$ in $\text{GrMod } A^G$ for every $i \in \mathbb{N}^+$.*

Proof. Since $A * G$ is Koszul such that $(A * G)_0 = kG$, we may write $A * G = T_{kG}((A * G)_1)/(\overline{R})$ for some $\overline{R} \subset (A * G)_1 \otimes_{kG} (A * G)_1$. By Lemma 2.6, the linear resolution of kG over $A * G$ is given by $(\overline{K}_i^i \otimes_{kG} A * G, \overline{d}^i)$ where

$$\overline{K}_i^i := \bigcap_{s+t+2=i} (A * G)_1^{\otimes_{kG} s} \otimes_{kG} \overline{R} \otimes_{kG} (A * G)_1^{\otimes_{kG} t} \subset (A * G)_1^{\otimes_{kG} i}$$

and \overline{d}^i is the restriction of the map

$$\begin{aligned} (v_1 * \epsilon) \otimes \cdots \otimes (v_{i-1} * \epsilon) \otimes (v_i * \epsilon) \otimes (a * g) &\mapsto (v_1 * \epsilon) \otimes \cdots \otimes (v_{i-1} * \epsilon) \otimes (v_i * \epsilon)(a * g) \\ &= (v_1 * \epsilon) \otimes \cdots \otimes (v_{i-1} * \epsilon) \otimes (v_i a * g) \\ (A * G)_1^{\otimes_{kG} i} \otimes_{kG} A * G &\longrightarrow (A * G)_1^{\otimes_{kG} i-1} \otimes_{kG} A * G \\ \varphi \uparrow \cong & \varphi \uparrow \cong \\ A_1^{\otimes_{kG} i} \otimes_k A * G &\longrightarrow A_1^{\otimes_{kG} i-1} \otimes_k A * G \\ v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i \otimes (a * g) &\mapsto v_1 \otimes \cdots \otimes v_{i-1} \otimes (v_i a * g) \end{aligned}$$

(using φ in Lemma 2.16, the above is a commutative diagram in $\text{GrMod } A * G$).

Since $(A * G)e \cong A$ as graded k - A bimodules by Lemma 2.15,

$$(A_1^{\otimes_{kG} i} \otimes_k A * G)e \cong A_1^{\otimes_{kG} i} \otimes_k (A * G)e \cong A_1^{\otimes_{kG} i} \otimes_k A$$

has a structure of a graded right A -module via ϕ in Lemma 2.15. By the commutative diagram

$$\begin{aligned} v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i \otimes (a * g)e &\mapsto v_1 \otimes \cdots \otimes v_{i-1} \otimes (v_i a * g)e \\ A_1^{\otimes_{kG} i} \otimes_k (A * G)e &\longrightarrow A_1^{\otimes_{kG} i-1} \otimes_k (A * G)e \\ \text{id} \otimes \phi \downarrow \cong & \text{id} \otimes \phi \downarrow \cong \\ A_1^{\otimes_{kG} i} \otimes_k A &\rightarrow A_1^{\otimes_{kG} i-1} \otimes_k A \\ v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i \otimes a &\mapsto v_1 \otimes \cdots \otimes v_{i-1} \otimes v_i a, \end{aligned}$$

$\overline{d}^i e$ can be viewed as a graded right A -module homomorphism.

Since $(-)e : \text{GrMod } A * G \rightarrow \text{GrMod } A^G$ is an exact functor and

$$(kG)e = (A * G)_0 e = ((A * G)e)_0 \cong A_0 = k$$

in $\text{GrMod } A^G$,

$$\cdots \xrightarrow{\overline{d}^3 e} (\overline{K}_2^2 \otimes_{kG} A * G)e \xrightarrow{\overline{d}^2 e} (\overline{K}_1^1 \otimes_{kG} A * G)e \xrightarrow{\overline{d}^1 e} (A * G)e \rightarrow (kG)e \cong k \rightarrow 0$$

is an exact sequence in $\text{GrMod } A^G$. Since it has a structure of a complex of graded right A -modules as we have seen above and every differential is of degree 1, it can be viewed as the linear resolution of k over A (that is, the image of the linear resolution of k over A under the functor $f_* : \text{GrMod } A \rightarrow \text{GrMod } A^G$ coincides with the above exact sequence). Since $(-)e : \text{GrMod } A * G \rightarrow \text{GrMod } A^G$ and $f_* : \text{GrMod } A \rightarrow \text{GrMod } A^G$ are exact functors, $(\Omega_{A * G}^i kG)e \cong f_* \Omega_A^i k$ in $\text{GrMod } A^G$ for every $i \in \mathbb{N}^+$. \square

Remark 2.18. Let $(\overline{K}^i := \overline{K}_i^i \otimes_{kG} A * G, \overline{d}^i)$ be the linear resolution of kG over $A * G$, and $(K^i := K_i^i \otimes_k A, d^i)$ the linear resolution of k over A . Although $(A * G)e$ has a graded left $A * G$ -module structure and a graded right A -module structure, it is not a graded $A * G$ - A bimodule, so

$$(-)e := - \otimes_{A * G} (A * G)e : \text{GrMod } A * G \rightarrow \text{GrMod } A$$

is not a functor, hence it is not clear if $(\overline{K}^i e, \overline{d}^i e)$ has a structure of a complex of graded right A -modules. We thank the referee for pointing this out. The above proof claims the following: Since $\overline{K}^i := \overline{K}_i^i \otimes_{kG} A * G \cong K_i^i \otimes_k A * G$ as graded right $A * G$ -modules, $\overline{K}^i e \cong (K_i^i \otimes_k A * G)e \cong K_i^i \otimes_k (A * G)e$ as graded right A^G -modules. On the other hand, since $(A * G)e \cong A$ as graded k - A bimodules by Lemma 2.15, $\overline{K}^i e \cong K_i^i \otimes_k (A * G)e \cong K_i^i \otimes_k A =: K^i$ has a structure of a graded right A -module, and we have shown above that the map $\overline{d}^i e$ coincides with the map d^i under these identifications.

2.3. AS-regular Algebras and Graded Frobenius Algebras.

Definition 2.19. [13, Definition 3.1] A locally finite graded algebra A is called AS-regular over A_0 of dimension d and of Gorenstein parameter ℓ if the following conditions are satisfied:

- (1) $\text{gldim } A_0 < \infty$.
- (2) $\text{gldim } A = d$.
- (3) A satisfies Gorenstein condition over A_0 , that is,

$$\underline{\text{Ext}}_A^i(A_0, A) \cong \begin{cases} D(A_0)(\ell) & \text{if } i = d, \\ 0 & \text{if } i \neq d \end{cases}$$

in $\text{GrMod } A$ and in $\text{GrMod } A^\circ$.

Remark 2.20. By [13, Corollary 3.7], a graded algebra A is AS-regular over A_0 of dimension d and of Gorenstein parameter ℓ if and only if A° is AS-regular over A_0° of dimension d and of Gorenstein parameter ℓ .

Lemma 2.21. [16, Corollary 3.6] *If S is a noetherian AS-regular algebra over k of dimension d and of Gorenstein parameter ℓ , and $G \leq \text{GrAut } S$ is a finite subgroup such that $\text{char } k$ does not divide $|G|$, then $S * G$ is a noetherian AS-regular algebra over $(S * G)_0 = kG$ of dimension d and of Gorenstein parameter ℓ .*

Definition 2.22. [11, Definition 12] A locally finite graded algebra A is called generalized AS-regular of dimension d if the following conditions are satisfied:

- (1) $A_i A_j = A_{i+j}$ for all $i, j \in \mathbb{N}$.
- (2) $\text{gldim } A = d$.
- (3) Every simple graded right A module has projective dimension d .
- (4) For every simple graded right A -module S , $\underline{\text{Ext}}_A^i(S, A) = 0$ for $i \neq d$.
- (5) The functors

$$\underline{\text{Ext}}_A^d(-, A) : \text{GrMod } A \longleftrightarrow \text{GrMod } A^\circ : \underline{\text{Ext}}_{A^\circ}^d(-, A)$$

induce a bijection between the set of all simple graded right A -modules and the set of all simple graded left A -modules.

Remark 2.23. In [11], Martinez-Villa called the algebra defined as above generalized Auslander regular. His definition requires $\text{sgldim } A = d$ instead of $\text{gldim } A = d$, but they are the same by [13, Proposition 2.7].

Lemma 2.24. *Every AS-regular algebra A generated by A_1 over A_0 is generalized AS-regular.*

Proof. Since A is generated by A_1 over A_0 , $A_i A_j = A_{i+j}$ for all $i, j \in \mathbb{N}$. By [13, Proposition 3.6], every simple graded right A module has projective dimension d . The rest of conditions follows from [13, Theorem 3.17]. \square

Definition 2.25. A locally finite graded algebra A is called graded Frobenius of Gorenstein parameter $-\ell$ if $D(A) \cong A(\ell)$ in $\text{GrMod } A$ (or equivalently in $\text{GrMod } A^o$).

Lemma 2.26. If A is a graded Frobenius algebra of Gorenstein parameter $-\ell$, and $G \leq \text{GrAut } A$ is a finite subgroup, then $A * G$ is a graded Frobenius algebra of Gorenstein parameter $-\ell$.

Proof. Let $\phi \in D(A)$ be the image of $1 \in A$ under an isomorphism $A(\ell) \rightarrow D(A)$ of graded right A -modules. Define $\Phi : A * G(\ell) \rightarrow D(A * G)$ by $(\Phi(a * g))(c * k) = \phi(ag(c))\delta_{gk}$ where

$$\delta_g = \begin{cases} 1 & \text{if } g = \epsilon \\ 0 & \text{if } g \neq \epsilon \end{cases} \quad (\epsilon \in G \text{ is the identity}). \text{ Since}$$

$$\begin{aligned} (\Phi((a * g)(b * h)))(c * k) &= (\Phi(ag(b) * gh))(c * k) \\ &= \phi(ag(b)(gh)(c))\delta_{(gh)k} \\ &= \phi(ag(b)g(h(c)))\delta_{ghk} \\ &= \phi(ag(bh(c)))\delta_{g(hk)} \\ &= (\Phi(a * g))(bh(c) * hk) \\ &= (\Phi(a * g))((b * h)(c * k)) \\ &= ((\Phi(a * b))(b * h))(c * k), \end{aligned}$$

Φ is a graded right $A * G$ -module homomorphism.

For $\sum_i a_i * g_i \in A * G$, if $\Phi(\sum_i a_i * g_i) = 0$, then

$$\Phi\left(\sum_i a_i * g_i\right)(c * g_j^{-1}) = \sum_i \phi(a_i g_i(c))\delta_{g_i g_j^{-1}} = \phi(a_j g_j(c)) = (\phi a_j)(g_j(c)) = 0$$

for all $c * g_j^{-1} \in A * G$, so $\phi a_j = 0 \in D(A)$ for all j . Since the map $A(\ell) \rightarrow D(A)$; $a \rightarrow \phi a$ is an isomorphism of graded right A -modules, $a_j = 0$ for all j , so Φ is injective. Since $\dim_k D(A * G) = \dim_k(A * G(\ell)) = (\dim_k A)|G| < \infty$, Φ is an isomorphism of graded right $A * G$ -modules. \square

Proposition 2.27. If S is an AS-regular Koszul algebra over k of dimension d and $G \leq \text{GrAut } S$ is a finite subgroup such that $\text{char } k$ does not divide $|G|$, then $S * G$ is a generalized AS-regular Koszul algebra and $(S * G)^! \cong G * S^!$ is a Frobenius Koszul algebra of Gorenstein parameter $-d$.

Proof. Since S is generalized AS-regular by Lemma 2.24, $S * G$ is also generalized AS-regular by [11, Lemma 13]. Since $S_0 = k$ is a simple graded right $S * G$ -module, S_0 has a graded projective resolution consisting of finitely generated graded projective right $S * G$ -modules by [13, Remark 3.16], so $S * G$ is Koszul and $(S * G)^! \cong G * S^!$ by Proposition 2.14. By [20, Proposition 5.10], $S^!$ is a Frobenius Koszul algebra of Gorenstein parameter $-d$, so $G * S^! \cong S^! * G^o$ is a graded Frobenius algebra of Gorenstein parameter $-d$ by Lemma 2.26. Since $G * S^!$ is finite dimensional (hence noetherian), $S_0^!$ has a graded projective resolution consisting of finitely generated graded projective right $G * S^!$ -modules, so $G * S^!$ is Koszul by Proposition 2.14. \square

2.4. Beilinson Algebras.

Definition 2.28. Let $r \in \mathbb{N}^+$.

- (1) For a graded vector space $V = \bigoplus_{i \in \mathbb{Z}} V_i$, we define the graded vector space $V^{(r)} := \bigoplus_{i \in \mathbb{Z}} V_{ri}$.
- (2) For a graded algebra A , the r -th Veronese algebra of A is the graded algebra $A^{(r)}$.

- (3) For a graded algebra A , the r -th quasi-Veronese algebra of A is the graded algebra defined by

$$A^{[r]} := \begin{pmatrix} A^{(r)} & A(1)^{(r)} & \cdots & A(r-1)^{(r)} \\ A(-1)^{(r)} & A^{(r)} & \cdots & A(r-2)^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ A(-r+1)^{(r)} & A(-r+2)^{(r)} & \cdots & A^{(r)} \end{pmatrix}$$

with the multiplication given by $(a_{ij})(b_{ij}) = (\sum_{k=0}^{r-1} a_{kj}b_{ik})$ for $a_{ij}, b_{ij} \in A(j-i)^{(r)}$.

Let A be a graded algebra and $r \in \mathbb{N}^+$. If a group G acts on A , then G naturally acts on $A^{[r]}$ by $g((a_{ij})) := (g(a_{ij}))$.

Lemma 2.29. *Let A be a graded algebra, and $G \leq \text{GrAut } A$ a subgroup. For $r \in \mathbb{N}^+$, $A^{[r]} * G \cong (A * G)^{[r]}$ as graded algebras.*

Proof. Since the map $A(j)^{(r)} \otimes_k kG \rightarrow (A * G)(j)^{(r)}$ defined by $(a_i) * g \mapsto (a_i * g)$ where $a_i \in A_{ri+j}$ is an isomorphism of graded vector spaces for each $j \in \mathbb{Z}$, the map

$$\phi : (A^{[r]} * G) \rightarrow (A * G)^{[r]}$$

defined by $\phi((a_{ij}) * g) = (a_{ij} * g)$ where $a_{ij} \in A(j-i)^{(r)}$ is an isomorphism of graded vector spaces. It is easy to check that ϕ is an isomorphism of graded algebras. \square

Definition 2.30. (1) The Beilinson algebra of an AS-regular algebra A over A_0 of Gorenstein parameter ℓ is defined by

$$\nabla A := (A^{[\ell]})_0 = \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-1} \\ 0 & A_0 & \cdots & A_{\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}.$$

- (2) The Beilinson algebra of a graded Frobenius algebra A of Gorenstein parameter $-\ell$ is defined by

$$\nabla A := (A^{[\ell]})_0 = \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-1} \\ 0 & A_0 & \cdots & A_{\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}.$$

Lemma 2.31. [16, Lemma 3.13] *If S is a noetherian AS-regular algebra over k and $G \leq \text{GrAut } S$ is a subgroup, then $\nabla(S * G) \cong (\nabla S) * G$ as algebras.*

Lemma 2.32. *If A is a graded Frobenius algebra and $G \leq \text{GrAut } A$ is a subgroup, then $\nabla(A * G) \cong (\nabla A) * G$ as algebras.*

Proof. If A is a graded Frobenius algebra of Gorenstein parameter $-\ell$, then $A * G$ is a graded Frobenius algebra of Gorenstein parameter $-\ell$ by Lemma 2.26, so

$$\nabla(A * G) = ((A * G)^{[\ell]})_0 \cong (A^{[\ell]} * G)_0 \cong (A^{[\ell]})_0 * G = (\nabla A) * G$$

as algebras by Lemma 2.29. \square

By the above lemmas, we can simply write $\nabla A * G$ for $(\nabla A) * G \cong \nabla(A * G)$.

Corollary 2.33. *If S is an AS-regular Koszul algebra over k of dimension d , and $G \leq \text{GrAut } S$ is a finite subgroup such that $\text{char } k$ does not divide $|G|$, then*

$$\nabla((S * G)^!) \cong \nabla(S^! * G^o) \cong \nabla(S^!) * G^o \cong G * \nabla(S^!)$$

as algebras.

3. DERIVED CATEGORIES OF NONCOMMUTATIVE PROJECTIVE SCHEMES

Let A be a graded right coherent algebra. Recall that the noncommutative projective scheme associated to A is defined by the quotient category $\text{tails } A := \text{grmod } A / \text{tors } A$ where $\text{grmod } A$ is the category of finitely presented graded right A -modules and $\text{tors } A$ is the full subcategory of $\text{grmod } A$ consisting of finite dimensional modules. In this section, we will find a finite dimensional algebra Λ such that $\mathcal{D}^b(\text{tails } A) \cong \mathcal{D}(\text{mod } \Lambda)$ when A is a “noncommutative quotient isolated singularity”.

3.1. BGG Correspondence. For a graded algebra A , we denote by $\text{grproj } A$ the full subcategory of $\text{grmod } A$ consisting of projective modules. For a triangulated category \mathcal{T} and a set T of objects in \mathcal{T} , we denote by $\text{thick } T$ the smallest full triangulated subcategory of \mathcal{T} containing T and closed under isomorphism and direct summands. The proof of the following lemma is left to the reader.

Lemma 3.1. *If A is a graded right coherent algebra, then the following hold:*

- (1) $\mathcal{D}^b(\text{grproj } A) = \text{thick}\{A(i)\}_{i \in \mathbb{Z}}$.
- (2) *If A_0 is finite dimensional and semisimple, then $\mathcal{D}^b(\text{tors } A) = \text{thick}\{A_0(i)\}_{i \in \mathbb{Z}}$.*

The existence of the Koszul equivalence below is essential in this paper. We refer to [4] for the definitions of the functor K and the categories $\mathcal{D}^\downarrow(A)$, $\mathcal{D}^\uparrow(A^!)$.

Lemma 3.2. [4, Theorem 2.12.1, Theorem 2.12.5] *If A is a Koszul algebra, then there exists an equivalence functor $K : \mathcal{D}^\downarrow(A) \rightarrow \mathcal{D}^\uparrow(A^!)$ such that $K(X[i](j)) = K(X)[i+j](-j)$, $K(A_0) = A^!$ and $K(D(A)) \cong A_0^!$.*

Note that the last isomorphism $K(D(A)) \cong A_0^!$ follows from the fact $D(A) \cong \underline{\text{Hom}}_{A_0}(A, A_0)$ by Lemma 2.2.

The next proposition, which is a generalization of the BGG correspondence, was essentially proved in [12], however, our definition of Koszul algebra is slightly more general than that given in [12], so we will repeat the proof to confirm the reader.

Proposition 3.3. (cf. [12, Proposition 4.1, Corollary 4.5]) *If A is a Frobenius Koszul algebra of Gorenstein parameter $-\ell$ such that $A^!$ is graded right coherent, then there exists an equivalence*

$$K : \mathcal{D}^b(\text{grmod } A) \rightarrow \mathcal{D}^b(\text{grmod } A^!)$$

of triangulated categories, which induces an equivalence

$$\overline{K} : \underline{\text{grmod}} A \rightarrow \mathcal{D}^b(\text{tails } A^!)$$

of triangulated categories.

Proof. Since $K(A_0(j)) \cong A^![j](-j) \in \text{thick}\{A^!(i)\}_{i \in \mathbb{Z}}$, and $K^{-1}(A^!(j)) \cong A_0[j](-j) \in \text{thick}\{A_0(i)\}_{i \in \mathbb{Z}}$ by Lemma 3.2, K restricts to an equivalence $K : \text{thick}\{A_0(i)\}_{i \in \mathbb{Z}} \rightarrow \text{thick}\{A^!(i)\}_{i \in \mathbb{Z}}$. Since A is finite dimensional, $\text{thick}\{A_0(i)\}_{i \in \mathbb{Z}} = \mathcal{D}^b(\text{tors } A) = \mathcal{D}^b(\text{grmod } A)$ and since $A^!$ has finite global dimension, $\text{thick}\{A^!(i)\}_{i \in \mathbb{Z}} = \mathcal{D}^b(\text{grproj } A^!) = \mathcal{D}^b(\text{grmod } A^!)$ by Lemma 3.1, so K induces an equivalence

$$K : \mathcal{D}^b(\text{grmod } A) \rightarrow \mathcal{D}^b(\text{grmod } A^!)$$

of triangulated categories.

Moreover, since A is graded Frobenius of Gorenstein parameter $-\ell$,

$$\begin{aligned} K(A(j)) &\cong K(D(A)(j - \ell)) \cong A_0^! [j - \ell](\ell - j) \in \text{thick}\{A_0^!(i)\}_{i \in \mathbb{Z}}, \text{ and} \\ K^{-1}(A_0^!(j)) &= D(A)[j](-j) \cong A[j](\ell - j) \in \text{thick}\{A(i)\}_{i \in \mathbb{Z}}, \end{aligned}$$

so K restricts to an equivalence functor $K : \text{thick}\{A(i)\}_{i \in \mathbb{Z}} \rightarrow \text{thick}\{A_0^!(i)\}_{i \in \mathbb{Z}}$. Since

$$\underline{\text{grmod}} A \cong \mathcal{D}^b(\text{grmod } A) / \mathcal{D}^b(\text{grproj } A) = \mathcal{D}^b(\text{grmod } A) / \text{thick}\{A(i)\}_{i \in \mathbb{Z}}, \text{ and}$$

$$\mathcal{D}^b(\text{tails } A^!) = \mathcal{D}^b(\text{grmod } A^! / \text{tors } A^!) \cong \mathcal{D}^b(\text{grmod } A^!) / \mathcal{D}^b(\text{tors } A^!) = \mathcal{D}^b(\text{grmod } A^!) / \text{thick}\{A_0^!(i)\}_{i \in \mathbb{Z}}$$

by Lemma 3.1 and [12, Theorem 4.4], K induces an equivalence

$$\overline{K} : \underline{\text{grmod}} A \rightarrow \mathcal{D}^b(\text{tails } A^!)$$

of triangulated categories. □

Let A be a Koszul algebra. Since

$$\begin{aligned} A_1 \otimes_{A_0} \text{Hom}_{A_0}(A_1, A_0) &\cong \text{Hom}_{A_0}(A_1, A_1 \otimes_{A_0} A_0) \cong \text{Hom}_{A_0}(A_1, A_1), \\ \text{Hom}_{A_0^o}(A_1, A_0) \otimes_{A_0} A_1 &\cong \text{Hom}_{A_0}(A_1, A_0 \otimes_{A_0} A_1) \cong \text{Hom}_{A_0^o}(A_1, A_1), \end{aligned}$$

there exist elements $v_\alpha \in A_1, {}^*v_\alpha \in \text{Hom}_{A_0}(A_1, A_0), v_\alpha^* \in \text{Hom}_{A_0^o}(A_1, A_0)$ such that $\sum v_\alpha \otimes {}^*v_\alpha$ corresponds to Id_{A_1} and $\sum v_\alpha^* \otimes v_\alpha$ corresponds to Id_{A_1} under the above isomorphisms, that is, for $w \in A_1$,

$$\begin{aligned} \sum v_\alpha ({}^*v_\alpha(w)) &= w, \\ \sum (v_\alpha^*(w)) v_\alpha &= w. \end{aligned}$$

We use the following lemma.

Lemma 3.4. *Let A be a Koszul algebra. For $V \in \text{Mod } A_0^o$, the map*

$$\text{Hom}_{A_0^o}(V, A_0) \otimes_{A_0} A^! \rightarrow \text{Hom}_{A_0^o}(V, A_0 \otimes_{A_0} A^!) \cong \text{Hom}_{A_0^o}(V, A^!)$$

defined by $\Phi(\phi \otimes a)(v) = \phi(v)a$ is an isomorphism in $\text{GrMod } A^!$.

Proposition 3.5. *Let A be a Koszul algebra. If $M \in \text{lin } A^!$, then $K(DE_{A^!}(M))$ is a linear resolution of M .*

Proof. Write $N := E_{A^!}(M) \in \text{GrMod } A^o$. By Lemma 2.2, we identify $D(N_i)$ with $\text{Hom}_{A_0^o}(N_i, A_0)$ as A_0 -modules. By the proof of [4, Theorem 2.12.1], $K(DN)$ is a complex

$$K(DN)^i = (DN)_{-i} \otimes_{A_0} A^!(-i) = D(E_{A^!}(M)_i) \otimes_{A_0} A^!(-i) = D \underline{\text{Ext}}_{A^!}^i(M, A_0) \otimes_{A_0} A^!(-i)$$

with differentials $d : D(N_i) \otimes_{A_0} A^!(-i) \rightarrow D(N_{i-1}) \otimes_{A_0} A^!(-i+1)$ given by

$$d(\phi \otimes a) = (-1)^i \sum \phi v_\alpha \otimes {}^*v_\alpha a.$$

By Lemma 3.4, we have a commutative diagram

$$\begin{array}{ccc} \text{Hom}_{A_0^o}(N_i, A_0) \otimes_{A_0} A^!(-i) & \xrightarrow{d} & \text{Hom}_{A_0^o}(N_{i-1}, A_0) \otimes_{A_0} A^!(-i+1) \\ \Phi \downarrow \cong & & \Phi \downarrow \cong \\ \text{Hom}_{A_0^o}(N_i, A^!)(-i) & \xrightarrow{d'} & \text{Hom}_{A_0^o}(N_{i-1}, A^!)(-i+1). \end{array}$$

For $v \in N_{i-1}$,

$$\begin{aligned} d'(\Phi(\phi \otimes a))(v) &= \Phi(d(\phi \otimes a))(v) = \Phi\left((-1)^i \sum \phi v_\alpha \otimes {}^*v_\alpha a\right)(v) \\ &= (-1)^i \sum (\phi v_\alpha)(v) {}^*v_\alpha a = (-1)^i \sum v_\alpha^* \phi(v_\alpha v) a \\ &= (-1)^i \sum v_\alpha^* \Phi(\phi \otimes a)(v_\alpha v) \end{aligned}$$

so $d'(f)(v) = (-1)^i \sum v_\alpha^* f(v_\alpha v)$ for $f \in \text{Hom}_{A_0^o}(N_i, A^!)$, hence $K(DE_{A^!}(M))$ is a linear resolution of M by [4, Proposition 2.14.5] and the comment after the proposition (see also the remark after [4, Definition 2.8.1]). \square

Lemma 3.6. *If A is a Koszul algebra and $M \in \text{lin } A^!$, then $DK^{-1}(M) \cong E_{A^!}(M)$ in $\mathcal{D}^b(\text{grmod } A^o)$.*

Proof. By Proposition 3.5, $K(DE_{A^!}(M)) \cong M$ in $\mathcal{D}^b(\text{grmod } A^!)$. Since $M \in \mathcal{D}^\uparrow(A^!)$ and $E_{A^!}(M)$ is locally finite, $DK^{-1}(M) \cong DDE_{A^!}(M) \cong E_{A^!}(M)$ in $\mathcal{D}^b(\text{grmod } A^o)$. \square

Lemma 3.7. *If A is a Koszul algebra and $M \in \text{lin } A$, then $E_A(\Omega^i M(i)) \cong E_A(M)_{\geq i}(i)$ in $\text{GrMod}(A^!)^o$ for all $i \in \mathbb{N}$.*

Proof. By Proposition 3.5, we have an exact sequence

$$0 \rightarrow \Omega^1 M \rightarrow D \underline{\text{Hom}}_A(M, A_0) \otimes_{A_0} A \rightarrow M \rightarrow 0$$

in $\text{GrMod } A$, which induces an exact sequence

$$0 \rightarrow \underline{\text{Hom}}_A(M, A_0) \rightarrow \underline{\text{Hom}}_A(D \underline{\text{Hom}}_A(M, A_0) \otimes_{A_0} A, A_0) \rightarrow \underline{\text{Hom}}_A(\Omega^1 M, A_0) \rightarrow \underline{\text{Ext}}_A^1(M, A_0) \rightarrow 0$$

and isomorphisms $\underline{\text{Ext}}_A^i(\Omega^1 M, A_0) \cong \underline{\text{Ext}}_A^{i+1}(M, A_0)$ for $i \geq 1$. Since

$$\begin{aligned} \underline{\text{Hom}}_A(D \underline{\text{Hom}}_A(M, A_0) \otimes_{A_0} A, A_0) &\cong \underline{\text{Hom}}_{A_0}(D \underline{\text{Hom}}_A(M, A_0), \underline{\text{Hom}}_A(A, A_0)) \\ &\cong \underline{\text{Hom}}_{A_0}(D \underline{\text{Hom}}_A(M, A_0), A_0) \\ &\cong D D \underline{\text{Hom}}_A(M, A_0) \\ &\cong \underline{\text{Hom}}_A(M, A_0) \end{aligned}$$

in $\text{GrMod}(A^!)^o$ by Lemma 2.2, $\underline{\text{Hom}}_A(\Omega^1 M, A_0) \cong \underline{\text{Ext}}_A^1(M, A_0)$. It follows that

$$\begin{aligned} E_A(\Omega^1 M(1)) &:= \bigoplus_{i=0}^{\infty} \underline{\text{Ext}}_A^i(\Omega^1 M(1), A_0) = \bigoplus_{i=0}^{\infty} \underline{\text{Ext}}_A^{i+1}(M, A_0) \\ &\cong \bigoplus_{i=1}^{\infty} \underline{\text{Ext}}_A^i(M, A_0)(1) \cong E_A(M)_{\geq 1}(1) \end{aligned}$$

in $\text{GrMod}(A^!)^o$. Since $\Omega^1 M(1) \in \text{lin } A$, the result follows by induction. \square

3.2. Tilting Objects.

Definition 3.8. Let \mathcal{T} be a triangulated category. An object $T \in \mathcal{T}$ is called tilting if

- (1) $\text{thick}(T) = \mathcal{T}$, and
- (2) $\text{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for all $i \neq 0$.

A tilting object plays an essential role in this paper due to the following result.

Theorem 3.9. (cf. [6, Theorem 2.2]) *Let \mathcal{T} be an algebraic Krull-Schmidt triangulated category and $T \in \mathcal{T}$ a tilting object. If $\text{gldim } \text{End}_{\mathcal{T}}(T) < \infty$, then $\mathcal{T} \cong \mathcal{D}^b(\text{mod } \text{End}_{\mathcal{T}}(T))$.*

Lemma 3.10. *If A is a graded Frobenius algebra of Gorenstein parameter $-\ell$, then $D(A_{\geq i}(i)) \cong A(\ell - i)/A(\ell - i)_{\geq 1}$ in $\text{grmod } A$.*

Proof. Since A is a graded Frobenius algebra, $D(A) \cong A(\ell)$ in $\text{grmod } A$. For each $i \in \mathbb{N}$, the inclusion $A_{\geq i} \rightarrow A$ in $\text{grmod } A^o$ induces a surjection $D(A) \rightarrow D(A_{\geq i})$ in $\text{grmod } A$. Since

$$D(A_{\geq i})_j = \text{Hom}_k((A_{\geq i})_{-j}, k) = \begin{cases} \text{Hom}_k(A_{-j}, k) = D(A)_j & \text{if } -j \geq i \\ 0 & \text{if } -j < i, \end{cases}$$

$$D(A_{\geq i}) \cong D(A)/D(A)_{\geq 1-i} \cong A(\ell)/A(\ell)_{\geq 1-i} = A/A_{\geq \ell+1-i}(\ell)$$

in $\text{grmod } A$, so

$$D(A_{\geq i}(i)) = D(A_{\geq i})(-i) \cong A/A_{\geq \ell+1-i}(\ell)(-i) = A(\ell-i)/A(\ell-i)_{\geq 1}$$

in $\text{grmod } A$. \square

Lemma 3.11. *If A is a graded Frobenius algebra of Gorenstein parameter $-\ell$ such that $\text{gldim } A_0 < \infty$, then*

- (1) $\bigoplus_{i=0}^{\ell-1} A(i)/A(i)_{\geq 1}$ is a tilting object for $\underline{\text{grmod}} A$,
- (2) $\text{End}_{\underline{\text{grmod}} A} \left(\bigoplus_{i=0}^{\ell-1} A(i)/A(i)_{\geq 1} \right) \cong \nabla A$, and
- (3) there is an equivalence $\underline{\text{grmod}} A \cong \mathcal{D}^b(\text{mod } \nabla A)$ of triangulated categories.

Proof. Since $A(i)/A(i)_{\geq 1} = A(i)$ are graded projective right A -modules for all $i \geq \ell$ and $A(i)/A(i)_{\geq 1} = 0$ for all $i < 0$, $U := \bigoplus_{i=0}^{\ell-1} A(i)/A(i)_{\geq 1}$ is a tilting object for $\underline{\text{grmod}} A$ by [22, Theorem 3.3 (2)].

If P is an indecomposable graded projective right A -module, then P is a direct summand of $A(-i)$ for some i , so $P = eA(-i)$ for some idempotent $e \in A$, hence $P_i \neq 0$. Since $D(P)$ is a direct summand of $D(A(-i)) \cong A(\ell+i)$ as a graded left A -module, $D(P)$ is an indecomposable graded projective left A -module, so $D(P_{\ell+i}) \cong D(P)_{-\ell-i} \neq 0$ by the same argument, hence $P_{\ell+i} \neq 0$. Since $U_i = 0$ for all $i > 0$ and $i \leq -\ell$, there is no projective direct summand in U , so $\text{End}_{\underline{\text{grmod}} A}(U) \cong \nabla A$ by the proof of [22, Lemma 3.9], and there is an equivalence $\underline{\text{grmod}} A \cong \mathcal{D}^b(\text{mod } \nabla A)$ of triangulated categories by [22, Theorem 3.11 (2)]. \square

Remark 3.12. The equivalence $\underline{\text{grmod}} A \cong \mathcal{D}^b(\text{mod } \nabla A)$ of triangulated categories also follows from [13, Theorem 4.22.3].

Let A be a graded right coherent algebra. Recall that $\pi : \text{grmod } A \rightarrow \text{tails } A$ denotes the quotient functor.

Theorem 3.13. *If A is a Frobenius Koszul algebra of Gorenstein parameter $-\ell$ such that $A^!$ is graded right coherent AS-regular over $A_0^!$, then*

- (1) $\bigoplus_{i=1}^{\ell} \pi \Omega^i A_0^!(i)$ is a tilting object for $\mathcal{D}^b(\text{tails } A^!)$,
- (2) $\text{End}_{A^!} \left(\bigoplus_{i=1}^{\ell} \pi \Omega^i A_0^!(i) \right) \cong \nabla A$, and
- (3) there is an equivalence $\mathcal{D}^b(\text{tails } A^!) \cong \mathcal{D}^b(\text{mod } \nabla A)$ of triangulated categories.

Proof. Since $\Omega^i A_0^!(i) \in \text{lin } A^!$,

$$\begin{aligned} DK^{-1} \left(\bigoplus_{i=1}^{\ell} \Omega^i A_0^!(i) \right) &\cong E_{A^!} \left(\bigoplus_{i=1}^{\ell} \Omega^i A_0^!(i) \right) \cong \bigoplus_{i=1}^{\ell} E_{A^!}(\Omega^i A_0^!(i)) \\ &\cong \bigoplus_{i=1}^{\ell} E_{A^!}(A_0^!)_{\geq i}(i) \cong \bigoplus_{i=1}^{\ell} A_{\geq i}(i) \end{aligned}$$

in $\mathcal{D}^b(\text{grmod } A^o)$ by Lemma 3.6 and Lemma 3.7, so

$$K^{-1} \left(\bigoplus_{i=1}^{\ell} \Omega^i A_0^!(i) \right) \cong D \left(\bigoplus_{i=1}^{\ell} A_{\geq i}(i) \right) \cong \bigoplus_{i=1}^{\ell} A(\ell-i)/A(\ell-i)_{\geq 1} \cong \bigoplus_{i=0}^{\ell-1} A(i)/A(i)_{\geq 1}$$

in $\text{grmod } A$ by Lemma 3.10. Since the functor K induces an equivalence $\overline{K} : \text{grmod } A \rightarrow \mathcal{D}^b(\text{tails } A^!)$ of triangulated categories by Proposition 3.3, $\bigoplus_{i=1}^{\ell} \pi \Omega^i A_0^!(i) = \overline{K}(\bigoplus_{i=0}^{\ell-1} A(i)/A(i)_{\geq 1})$ is a tilting object for $\mathcal{D}^b(\text{tails } A^!)$ such that

$$\text{End}_{A^!} \left(\bigoplus_{i=1}^{\ell} \pi \Omega^i A_0^!(i) \right) \cong \text{End}_{\text{grmod } A} \left(\bigoplus_{i=0}^{\ell-1} A(i)/A(i)_{\geq 1} \right) \cong \nabla A$$

and there is an equivalence $\mathcal{D}^b(\text{tails } A^!) \cong \text{grmod } A \cong \mathcal{D}^b(\text{mod } \nabla A)$ of triangulated categories by Lemma 3.11. \square

Remark 3.14. If A is a noetherian AS-regular algebra over A_0 of dimension d , then the noncommutative projective scheme $Y := \text{Proj}_{nc} A$ associated to A is regarded as a quantum projective space of dimension $d-1$ over $\text{Spec}_{nc} A_0$. If A is Koszul, then the i -th sheaf of differentials on Y is defined by $\Omega_Y^i = \pi \Omega_{A^!}^{i+1} A_0 \in \text{tails } A$ for $i \in \mathbb{N}$ in [14, Definition 5.7]. By the above theorem, $\bigoplus_{i=0}^{d-1} \Omega_Y^i(i) = (\bigoplus_{i=1}^d \pi \Omega_A^i A_0(i))(-1)$ is a tilting object for $\mathcal{D}^b(\text{tails } A)$. In fact, via the equivalence $\overline{K} : \text{grmod } A^! \rightarrow \mathcal{D}^b(\text{tails } A)$, the tilting object $\bigoplus_{i=0}^{d-1} \Omega_Y^i(i)$ for $\mathcal{D}^b(\text{tails } A)$ corresponds to the tilting object $\bigoplus_{i=1}^d A_{\geq i}^!(i)$ for $\text{grmod } A^!$.

Corollary 3.15. *If S is a noetherian AS-regular Koszul algebra over k of dimension d and $G \leq \text{GrAut } S$ is a finite subgroup such that $\text{char } k$ does not divide $|G|$, then*

- (1) $\bigoplus_{i=1}^d \pi \Omega_{S * G}^i kG(i)$ is a tilting object for $\mathcal{D}^b(\text{tails } S * G)$,
- (2) $\text{End}_{S * G} \left(\bigoplus_{i=1}^d \pi \Omega_{S * G}^i kG(i) \right) \cong G * \nabla(S^!)$, and
- (3) *there is an equivalence $\mathcal{D}^b(\text{tails } S * G) \cong \mathcal{D}^b(\text{mod } G * \nabla(S^!))$ of triangulated categories.*

Proof. Since $S * G$ is a noetherian AS-regular Koszul algebra over $(S * G)_0 = kG$ of dimension d , and $(S * G)^!$ is a Frobenius Koszul algebra of Gorenstein parameter $-d$ by Lemma 2.21 and Proposition 2.27, $\bigoplus_{i=1}^d \pi \Omega_{S * G}^i kG(i)$ is a tilting object for $\mathcal{D}^b(\text{tails } S * G)$ such that

$$\text{End}_{S * G} \left(\bigoplus_{i=1}^d \pi \Omega_{S * G}^i kG(i) \right) \cong \nabla((S * G)^!) \cong G * \nabla(S^!),$$

and there is an equivalence $\mathcal{D}^b(\text{tails } S * G) \cong \mathcal{D}^b(\text{mod } G * \nabla(S^!))$ of triangulated categories by Theorem 3.13 and Corollary 2.33. \square

Lemma 3.16. *If S is an AS-regular Koszul algebra over k of dimension d , $G \leq \text{GrAut } S$ is a finite subgroup such that $\text{char } k$ does not divide $|G|$, and $e = \frac{1}{|G|} \sum_{g \in G} g \in kG \subset S * G$ so that $e(S * G)e \cong S^G$, then $(\Omega_{S * G}^i kG)e \cong f_* \Omega_S^i k$ in $\text{GrMod } S^G$ for every $i \in \mathbb{N}^+$ where $f : S^G \rightarrow S$ is the inclusion.*

Proof. Since $S * G$ is Koszul by Proposition 2.27, the result follows from Theorem 2.17. \square

Let A be a graded right coherent algebra. Recall our notations that $\mathcal{M} = \pi M \in \text{tails } A$ is the image of $M \in \text{grmod } A$, and $\text{Hom}_A(\mathcal{M}, \mathcal{N}) := \text{Hom}_{\text{tails } A}(\pi M, \pi N)$.

Lemma 3.17. *Let A be a right noetherian graded algebra and $e \in A$ an idempotent such that eAe is a right noetherian graded algebra. If $Ae \in \text{grmod } eAe$, and*

$$(-)e : \text{tails } A \rightarrow \text{tails } eAe$$

is an equivalence functor, then $A/(e)$ is a torsion A -module.

Proof. Let $F : \text{grmod } A/(e) \rightarrow \text{grmod } A$ be the restriction functor induced by the natural epimorphism from A to $A/(e)$. It is easy to check that F is faithful. Let $G := (-)e : \text{grmod } A \rightarrow \text{grmod } eAe$. Since F and G preserve torsion modules, these functors induce the functors

$$\text{tails } A/(e) \xrightarrow{\overline{F}} \text{tails } A \xrightarrow{\overline{G}=(-)e} \text{tails } eAe.$$

Since F is faithful,

$$\begin{aligned} \text{Hom}_{A/(e)}(\mathcal{M}, \mathcal{N}) &\cong \lim_{n \rightarrow \infty} \text{Hom}_{A/(e)}(M_{\geq n}, N) \\ &\hookrightarrow \lim_{n \rightarrow \infty} \text{Hom}_A(M_{\geq n}, N) \\ &\cong \text{Hom}_A(\mathcal{M}, \mathcal{N}) \cong \text{Hom}_A(\overline{F}(\mathcal{M}), \overline{F}(\mathcal{N})), \end{aligned}$$

so \overline{F} is also faithful. Moreover, for any $\mathcal{M} \in \text{tails } A/(e)$, it is easy to see that $\overline{GF}(\mathcal{M}) = 0$. Since $\overline{G} = (-)e$ is an equivalence functor, $\overline{F}(\mathcal{M}) = 0$. It follows from the faithfulness of \overline{F} that $\text{Hom}_{A/(e)}(\mathcal{M}, \mathcal{M}) \hookrightarrow \text{Hom}_A(\overline{F}(\mathcal{M}), \overline{F}(\mathcal{M})) = 0$, so the identity morphism for \mathcal{M} is zero. Thus $\mathcal{M} = 0$ for any $\mathcal{M} \in \text{tails } A/(e)$. We see that $A/(e) \in \text{tors } A/(e)$ and hence $A/(e) \in \text{tors } A$. \square

Proposition 3.18. *Let A be a right noetherian connected graded algebra, $G \leq \text{GrAut } A$ a finite subgroup such that $\text{char } k$ does not divide $|G|$, and $e = \frac{1}{|G|} \sum_{g \in G} g \in kG \subset A * G$. Then the following are equivalent:*

- (1) $A * G/(e)$ is finite dimensional over k .
- (2) $(-)e : \text{tails } A * G \rightarrow \text{tails } A^G$ is an equivalence functor.

Proof. This follows from [16, Theorem 2.13] and Lemma 3.17. \square

Remark 3.19. The above equivalent conditions are closely related to the condition that S^G is a graded isolated singularity [16, Theorem 3.10]. The condition (2) is called “ample group action” in [16].

Theorem 3.20. *Let S be a noetherian AS-regular Koszul algebra over k of dimension d , $G \leq \text{GrAut } S$ a finite subgroup such that $\text{char } k$ does not divide $|G|$, and $f : S^G \rightarrow S$ the inclusion map. If $S * G/(e)$ is finite dimensional over k , then*

- (1) $\bigoplus_{i=1}^d \pi f_* \Omega_S^i k(i)$ is a tilting object for $\mathcal{D}^b(\text{tails } S^G)$,
- (2) $\text{End}_{S^G} \left(\bigoplus_{i=1}^d \pi f_* \Omega_S^i k(i) \right) \cong G * \nabla(S^!)$, and
- (3) there is an equivalence $\mathcal{D}^b(\text{tails } S^G) \cong \mathcal{D}^b(\text{mod } G * \nabla(S^!))$ of triangulated categories.

Proof. Since $S * G/(e)$ is finite dimensional over k , $(-)e : \text{tails } S * G \rightarrow \text{tails } S^G$ is an equivalence functor by Proposition 3.18. Since $\bigoplus_{i=1}^d \pi \Omega_{S * G}^i kG(i)$ is a tilting object for $\mathcal{D}^b(\text{tails } S * G)$ by Corollary 3.15, and $(\bigoplus_{i=1}^d \pi \Omega_{S * G}^i kG(i))e \cong \bigoplus_{i=1}^d \pi f_* \Omega_S^i k(i)$ in $\text{tails } S^G$ by Lemma 3.16, $\bigoplus_{i=1}^d \pi f_* \Omega_S^i k(i)$ is a tilting object for $\mathcal{D}^b(\text{tails } S^G)$ such that

$$\text{End}_{S^G} \left(\bigoplus_{i=1}^d \pi f_* \Omega_S^i k(i) \right) \cong \text{End}_{S * G} \left(\bigoplus_{i=1}^d \pi \Omega_{S * G}^i kG(i) \right) \cong G * \nabla(S^!),$$

and there is an equivalence $\mathcal{D}^b(\text{tails } S^G) \cong \mathcal{D}^b(\text{tails } S * G) \cong \mathcal{D}^b(\text{mod } G * \nabla(S^!))$ of triangulated categories by Corollary 3.15. \square

Remark 3.21. In the setting of the above theorem, the inclusion map $f : S^G \rightarrow S$ induces a functor $f_* : \text{tails } S \rightarrow \text{tails } S^G$. Since $S * G/(e)$ is finite dimensional over k , the functor $(-)e : \text{GrMod } S * G \rightarrow \text{GrMod } S^G$ induces an equivalence $(-)e : \text{tails } S * G \rightarrow \text{tails } S^G$ by Proposition 3.18. Write $X = \text{Proj}_{nc} S$ and $Y = \text{Proj}_{nc} S * G$. It is easy to see that $\Omega_X^{d-1}(d-1), \dots, \Omega_X^1(1), \Omega_X^0$

is a full strong exceptional sequence for $\mathcal{D}^b(\text{tails } S)$ (cf. [14, Proposition 4.8, Theorem 5.11]). Since $\text{End}_{S^G}(f_*\Omega_X^i(i)) \cong kG$, $f_*\Omega_X^{d-1}(d-1), \dots, f_*\Omega_X^1(1), f_*\Omega_X^0$ is no longer an exceptional sequence for $\mathcal{D}^b(\text{tails } S^G)$ in the usual sense unless G is trivial, however, it is a full strong exceptional sequence “over kG ”. In fact, $\bigoplus_{i=0}^{d-1} \Omega_Y^i(i)$ is a tilting object for $\mathcal{D}^b(\text{tails } S * G)$ by Remark 3.14, so

$$\bigoplus_{i=0}^{d-1} f_*\Omega_X^i(i) = \left(\bigoplus_{i=1}^d \pi f_*\Omega_S^i k(i) \right) (-1) \cong \left(\bigoplus_{i=1}^d \pi \Omega_{S*G}^i kG(i) \right) e(-1) = \left(\bigoplus_{i=0}^{d-1} \Omega_Y^i(i) \right) e$$

is a tilting object for $\mathcal{D}^b(\text{tails } S^G)$ as in the above theorem. If kG is a finite direct product of k , then we may obtain a full strong exceptional sequence for $\mathcal{D}^b(\text{tails } S^G)$ by replacing each $f_*\Omega_X^i(i)$ with its set of indecomposable direct summands in a suitable order. If kG is not a finite direct product of k , then we may still form a full strong exceptional sequence by deleting isomorphic indecomposable direct summands (cf. [6, Corollary 2.12]). We thank the referee for pointing this out.

4. STABLE CATEGORIES OF MAXIMAL COHEN-MACAULAY MODULES

Let A be a noetherian AS-Gorenstein algebra over k . Then the graded singularity category is defined by the Verdier localization $\mathcal{D}_{\text{Sg}}^{\text{gr}}(A) := \mathcal{D}^b(\text{grmod } A) / \mathcal{D}^b(\text{grproj } A)$. We denote the localization functor by $v : \mathcal{D}^b(\text{grmod } A) \rightarrow \mathcal{D}_{\text{Sg}}^{\text{gr}}(A)$. By [5], $\mathcal{D}_{\text{Sg}}^{\text{gr}}(A) \cong \underline{\text{CM}}^{\mathbb{Z}} A$ the stable category of graded maximal Cohen-Macaulay modules over A . In this section, we will find a finite dimensional algebra Γ such that $\underline{\text{CM}}^{\mathbb{Z}} A \cong \mathcal{D}(\text{mod } \Gamma)$ when A is a “noncommutative quotient isolated singularity”.

4.1. Tilting Objects. Let A be a noetherian AS-Gorenstein algebra over k of Gorenstein parameter ℓ . If $\ell > 0$, then there exists the embedding $\Phi := \Phi_0 : \mathcal{D}_{\text{Sg}}^{\text{gr}}(A) \rightarrow \mathcal{D}^b(\text{tails } A)$ by Orlov [17]. Unfortunately, $\Phi v \neq \pi$, but we have the following result.

Lemma 4.1. *Let A be a noetherian AS-Gorenstein algebra over k of positive Gorenstein parameter, and $M \in \text{grmod } A$, If $M_{\geq 0} = M$ and $\text{Hom}_A(M, A(i)) = 0$ for all $i \leq 0$, then $\Phi(vM) \cong \pi M$.*

Proof. This follows from the proof of [1, Theorem 4.3]. \square

Let A be a right noetherian graded algebra. For $M \in \text{grmod } A$, we define

$$\text{depth } M = \inf \{i \mid \underline{\text{Ext}}_A^i(A_0, M) \neq 0\}.$$

If A is a right noetherian graded algebra such that A_0 is a finite dimensional semisimple algebra (eg. A is Koszul) and $M \in \text{grmod } A$, then one can show that

$$\underline{\text{Ext}}_A^i(T, M) = 0$$

for any $T \in \text{tors } A$ and any $i < \text{depth } M$.

Lemma 4.2. *Let A be a right noetherian graded algebra such that A_0 is a finite dimensional semisimple algebra and $M, N \in \text{grmod } A$. If $\text{depth } N \geq 2$, then the natural map*

$$\underline{\text{Hom}}_A(M, N) \rightarrow \underline{\text{Hom}}_A(\mathcal{M}, \mathcal{N})$$

is an isomorphism of vector spaces.

Proof. The proof is same as that of [15, Lemma 2.9]. \square

Setting 4.3. We fix the following setting:

- (1) S is a noetherian AS-regular Koszul algebra over k of dimension $d \geq 2$ (so that the Gorenstein parameter is $\ell = d$).
- (2) $G \leq \text{GrAut } S$ is a finite subgroup such that $\text{char } k$ does not divide $|G|$.
- (3) $e = \frac{1}{|G|} \sum_{g \in G} g \in kG \subset S * G$ and $e' = 1 - e \in kG \subset S * G$.

- (4) S^G is AS-Gorenstein and $S * G / (e)$ is finite dimensional over k (so that S^G is a “Gorenstein isolated singularity”).
- (5) $U = \bigoplus_{i=1}^d \Omega_{S * G}^i kG(i)$. (Note that U is a graded kG - $S * G$ bimodule by Remark 2.11.)
- (6) (K^\bullet, d) is a linear resolution of kG over S .

Let $f : S^G \rightarrow S$ be the inclusion map. In Setting 4.3, $\pi Ue \cong \pi(\bigoplus_{i=1}^d f_* \Omega_S^i k(i)) \in \text{tails } S^G$ is a tilting object for $\mathcal{D}^b(\text{tails } S^G)$ such that $\text{End}_{S^G}(\pi Ue) \cong G * \nabla(S^!)$ by Lemma 3.16 and Theorem 3.20.

Lemma 4.4. *In Setting 4.3, $\Phi(ve'Ue) = \pi e'Ue$.*

Proof. Since $S * G$ is Koszul, $\Omega_{S * G}^i kG$ is generated in degree i , so $\Omega_{S * G}^i kG(i)$ is generated in degree 0 for every $1 \leq i \leq d$, hence there exists a surjective homomorphism $\bigoplus e'(S * G) \rightarrow e'U \rightarrow 0$. It follows that $(e'Ue)_{\geq 0} = e'Ue \in \text{grmod } S^G$, so it is enough to show that $\text{Hom}_{S^G}(e'Ue, S^G(i)) = 0$ for all $i \leq 0$ by Lemma 4.1. Since S^G is an AS-Gorenstein algebra of dimension $d \geq 2$, it follows that $\text{depth}_{S^G}(S^G) \geq 2$. Since $S * G$ is an AS-regular algebra over kG of dimension $d \geq 2$ by Lemma 2.21, $\text{depth}_{S * G} S = \text{depth}_{S * G} e(S * G) = \text{depth}_{S * G}(S * G) \geq 2$. By Lemma 4.2,

$$\begin{aligned}
 \text{Hom}_{S^G}(e'Ue, S^G(i)) &\cong \text{Hom}_{S^G}(\pi e'Ue, \pi S^G(i)) \\
 &\cong \text{Hom}_{S * G}(\pi e'U, \pi S(i)) \\
 &\cong \text{Hom}_{S * G}(e'U, e(S * G)(i)) \\
 &\hookrightarrow \text{Hom}_{S * G}\left(\bigoplus e'(S * G), e(S * G)(i)\right) \\
 &\cong \bigoplus (e(S * G)e')_i.
 \end{aligned}$$

Since $(e(S * G)e')_0 = ekGe' = 0$, $(e(S * G)e')_i = 0$ for all $i \leq 0$, hence the result. \square

Lemma 4.5. *In Setting 4.3, $e'(S * G)e(i) \in \text{thick}(e'Ue)$ in $\mathcal{D}^b(\text{grmod } S^G)$ for all $i \in \mathbb{Z}$.*

Proof. Recall that $e'Ue = \bigoplus_{i=1}^d e'(\Omega_{S * G}^i kG)e(i)$ so that $e'(\Omega_{S * G}^i kG)e(i) \in \text{thick}(e'Ue)$ for $1 \leq i \leq d$. The (truncated) linear resolution of kG over $S * G$

$$0 \rightarrow \Omega_{S * G}^i kG \rightarrow K^{i-1} \rightarrow K^{i-2} \rightarrow \cdots \rightarrow K^1 \rightarrow S * G \rightarrow kG \rightarrow 0$$

induce a long exact sequence

$$0 \rightarrow e'(\Omega_{S * G}^i kG)e(i) \rightarrow e'K^{i-1}e(i) \rightarrow e'K^{i-2}e(i) \rightarrow \cdots \rightarrow e'K^1e(i) \rightarrow e'(S * G)e(i) \rightarrow e'kGe(i) = 0$$

for $1 \leq i \leq d$. It follows that $e'(S * G)e(1) \cong e'K^0e(1) \cong e'(\Omega_{S * G}^1 kG)e(1) \in \text{thick}(e'Ue)$. Since K^i is a graded projective right $S * G$ -module generated in degree i , $K^i \in \text{add}(S * G(-i))$. It follows that $e'K^1e(2) \in \text{add}(e'(S * G)e(1))$, so $e'(S * G)e(2) \cong e'(\Omega_{S * G}^1 kG)e(2) \in \text{thick}(e'Ue)$. By induction, $e'(S * G)e(i) \in \text{thick}(e'Ue)$ for $1 \leq i \leq d$.

Since $K^d = S * G(-d)$, we have a long exact sequence

$$0 \rightarrow e'(S * G)e(i) \rightarrow e'K^{d-1}e(i+d) \rightarrow \cdots \rightarrow e'K^1e(i+d) \rightarrow e'(S * G)e(i+d) \rightarrow 0$$

for each $i \in \mathbb{Z}$. Since $e'K^i e(j+i) \in \text{add}(e'(S * G)e(j)) \subset \text{thick}(e'Ue)$ for all $0 \leq i \leq d, 1 \leq j \leq d$, we can show that $e'(S * G)e(i) \in \text{thick}(e'Ue)$ for all $i \in \mathbb{Z}$ by induction. \square

Lemma 4.6. *In Setting 4.3, $v(S * G)e(i) \in \text{thick}(ve'Ue)$ in $\mathcal{D}_{\text{Sg}}^{\text{gr}}(S^G)$ for all $i \in \mathbb{Z}$.*

Proof. Since $e(S * G)e(i) \cong S^G(i)$ in $\text{grmod } S^G$, $v(S * G)e(i) = ve(S * G)e(i) \oplus ve'(S * G)e(i) = ve'(S * G)e(i)$ in $\mathcal{D}_{\text{Sg}}^{\text{gr}}(S^G)$. Since $v : \mathcal{D}^b(\text{grmod } S^G) \rightarrow \mathcal{D}_{\text{Sg}}^{\text{gr}}(S^G)$ is a localization functor,

$$v(S * G)e(i) = ve'(S * G)e(i) \in \text{thick}(ve'Ue)$$

for all $i \in \mathbb{Z}$ by Lemma 4.5. \square

Lemma 4.7. *In Setting 4.3, $\text{thick}\{v(S * G)e(i) \mid i \in \mathbb{Z}\} = \mathcal{D}_{\text{Sg}}^{\text{gr}}(S^G)$.*

Proof. For any object $X \in \mathcal{D}_{\text{Sg}}^{\text{gr}}(S^G)$, there exists $M \in \text{CM}^{\mathbb{Z}}(S^G)$ such that $X \cong vM$ by Buchweitz [5]. Since S is a $(d-1)$ -cluster tilting object in $\text{CM}^{\mathbb{Z}}(S^G)$ by [16, Theorem 3.15], we have an exact sequence

$$0 \rightarrow T_{d-1} \rightarrow \cdots \rightarrow T_1 \rightarrow T_0 \rightarrow M \rightarrow 0$$

in $\text{grmod } S^G$ where $T_i \in \text{add}\{S(i) \mid i \in \mathbb{Z}\}$ as in the proof of [21, Theorem 5.10], so it follows that $M \in \text{thick}\{S(i) \mid i \in \mathbb{Z}\}$ in $\mathcal{D}(\text{grmod } S^G)$. Hence

$$X \cong vM \in \text{thick}\{vS(i) \mid i \in \mathbb{Z}\} = \text{thick}\{v(S * G)e(i) \mid i \in \mathbb{Z}\}$$

in $\mathcal{D}_{\text{Sg}}^{\text{gr}}(S^G)$. \square

Proposition 4.8. *In Setting 4.3, $ve'Ue$ is a tilting object for $\mathcal{D}_{\text{Sg}}^{\text{gr}}(S^G)$.*

Proof. Since $\pi e'Ue$ is a direct summand of a tilting object πUe for $\mathcal{D}^b(\text{tails } S^G)$,

$$\text{Hom}_{\mathcal{D}_{\text{Sg}}^{\text{gr}}(S^G)}(ve'Ue, ve'Ue[i]) \cong \text{Hom}_{S^G}(\pi e'Ue, \pi e'Ue[i]) \subset \text{Hom}_{S^G}(\pi Ue, \pi Ue[i]) = 0$$

for all $i \neq 0$ by Lemma 4.4. Since $v(S * G)e(i) \in \text{thick}(ve'Ue)$ for all $i \in \mathbb{Z}$ by Lemma 4.6, we have that $\text{thick}(ve'Ue) = \text{thick}\{v(S * G)e(i) \mid i \in \mathbb{Z}\} = \mathcal{D}_{\text{Sg}}^{\text{gr}}(S^G)$ by Lemma 4.7, hence the result. \square

Lemma 4.9. *In Setting 4.3, $e'Ue \in \text{CM}^{\mathbb{Z}}(S^G)$.*

Proof. The (truncated) linear resolution of kG over $S * G$ induce a long exact sequence

$$0 \rightarrow e'(\Omega_{S * G}^i kG)e \rightarrow e'K^{i-1}e \rightarrow e'K^{i-2}e \rightarrow \cdots \rightarrow e'K^1e \rightarrow e'(S * G)e \rightarrow e'kGe = 0$$

for $1 \leq i \leq d$. It follows that

$$e'(\Omega_{S * G}^1 kG)e \cong e'(S * G)e, \text{ and } 0 \rightarrow e'(\Omega_{S * G}^i kG)e \rightarrow e'K^{i-1}e \rightarrow e'(\Omega_{S * G}^{i-1} kG)e \rightarrow 0 \quad (4.1)$$

for $2 \leq i \leq d$. Since $e'(S * G)e, e'K^{i-1}e \in \text{add}\{(S * G)e(i) \mid i \in \mathbb{Z}\} = \text{add}\{S(i) \mid i \in \mathbb{Z}\}$, these are graded maximal Cohen-Macaulay over S^G , so it follows from (4.1) that $e'(\Omega_{S * G}^i kG)e \in \text{CM}^{\mathbb{Z}}(S^G)$ for each $1 \leq i \leq d$. Hence we have $e'Ue = \bigoplus_{i=1}^d e'(\Omega_{S * G}^i kG)e(i) \in \text{CM}^{\mathbb{Z}}(S^G)$. \square

Theorem 4.10. *In Setting 4.3, $e'Ue$ is a tilting object for $\underline{\text{CM}}^{\mathbb{Z}}(S^G)$.*

Proof. Under the equivalence functor $\underline{\text{CM}}^{\mathbb{Z}}(S^G) \rightarrow \mathcal{D}_{\text{Sg}}^{\text{gr}}(S^G)$ given by Buchweitz [5], $e'Ue$ corresponds to $ve'Ue$ by Lemma 4.9. Thus Proposition 4.8 implies the result. \square

4.2. Endomorphism Algebras. In this subsection, we calculate the endomorphism algebra of the tilting object found in the previous subsection.

Lemma 4.11. *Let \mathcal{C} be an abelian category, and $M \in \mathcal{C}$ an object. For an idempotent element $e \in \text{End}_{\mathcal{C}}(M)$, $\text{End}_{\mathcal{C}}(eM) \cong e \text{End}_{\mathcal{C}}(M)e$ as rings where $eM := \text{Im } e$.*

Let A be a Koszul algebra over A_0 and $e \in A$ an idempotent. Since $e^2 = e$, it follows that $e \in A_0$. As stated in Remark 2.11, the linear resolution (K^\bullet, d^\bullet) of A_0 is a complex of graded A_0 - A bimodules. Since K^i is a graded projective right A -module generated in degree i , eK^i is also a graded projective right A -module generated in degree i . Since the functor $e(-) = eA_0 \otimes_{A_0} - : \text{GrMod } A_0 \otimes A \rightarrow \text{GrMod } A$ is exact, (eK^\bullet, ed^\bullet) is a linear resolution of eA_0 , so $e\Omega^i A_0 := e \text{Im } d^i \cong \text{Im } ed^i =: \Omega^i(eA_0)$. We define the map $e_i : \Omega^i A_0(i) \rightarrow \Omega^i A_0(i)$ induced by the left multiplication by e .

Lemma 4.12. *Let A be an AS-regular Koszul algebra over A_0 of dimension d , and $U := \bigoplus_{i=1}^d \Omega^i A_0(i)$. For an idempotent $e \in A$, $\text{End}_A(\pi eU) \cong \pi \bar{e} \text{End}_A(\pi U) \pi \bar{e}$ as rings where $\bar{e} = \bigoplus_{i=1}^d e_i \in \text{End}_A(U)$.*

Proof. Since $\text{Im } e_i = e\Omega^i A_0(i)$,

$$\text{Im } \bar{e} \cong \bigoplus_{i=1}^d \text{Im } e_i \cong \bigoplus_{i=1}^d e\Omega^i A_0(i) =: eU.$$

Since $\pi : \text{grmod } A \rightarrow \text{tails } A$ is an exact functor, $\text{Im}(\pi\bar{e}) \cong \pi \text{Im } \bar{e} \cong \pi eU$. Since $e_i \in \text{End}_A(\Omega^i A_0(i))$ is an idempotent, $\pi\bar{e} \in \text{End}_A(\pi eU)$ is an idempotent, so $\text{End}_A(\pi eU) \cong \pi\bar{e} \text{End}_A(\pi eU) \pi\bar{e}$ as rings by Lemma 4.11. \square

Lemma 4.13. *Let A be a graded Frobenius algebra of Gorenstein parameter $-\ell$ and $a \in A_i$. If $ba = 0$ for every $b \in A_{\ell-i}$, then $a = 0$.*

Proof. Since A is a graded Frobenius algebra of Gorenstein parameter $-\ell$, there exists an isomorphism $\Phi : A \rightarrow (DA)(-\ell)$ of graded right A -modules. If $\phi := \Phi(1) \in (DA)(-\ell)_0 = (DA)_{-\ell}$, then the map $\langle -, - \rangle : A \times A \rightarrow k$ defined by $\langle x, y \rangle = \phi(xy)$ is a nondegenerate (associative) bilinear form. For $x = \sum_{j=0}^{\ell} x_j \in A$, if $j = \ell - i$, then $\langle x_j, a \rangle = \phi(x_j a) = \phi(0) = 0$ by assumption and, if $j \neq \ell - i$, then $x_j a \in A_{i+j} \neq A_{\ell}$, so $\langle x_j, a \rangle = \phi(x_j a) = 0$ since $\phi \in (DA)_{-\ell}$, hence $\langle x, a \rangle = \langle \sum_{j=0}^{\ell} x_j, a \rangle = \sum_{j=0}^{\ell} \langle x_j, a \rangle = 0$. Since $\langle -, - \rangle$ is nondegenerate, $a = 0$. \square

Lemma 4.14. *Let A be a graded Frobenius algebra of Gorenstein parameter $-\ell$. For $0 \leq i, j \leq \ell$, the map $A_{j-i} \rightarrow \text{Hom}_{A^o}(A_{\geq i}(i), A_{\geq j}(j))$ defined by $a \mapsto \cdot a$ is an isomorphism.*

Proof. The exact sequences

$$0 \rightarrow A_{\geq i} \rightarrow A \rightarrow A/A_{\geq i} \rightarrow 0, \quad 0 \rightarrow A_{\geq j} \rightarrow A \rightarrow A/A_{\geq j} \rightarrow 0$$

in $\text{GrMod } A$ induce exact sequences

$$\begin{aligned} 0 \rightarrow \text{Hom}_{A^o}(A/A_{\geq i}(i), A(j)) &\rightarrow \text{Hom}_{A^o}(A(i), A(j)) \rightarrow \text{Hom}_{A^o}(A_{\geq i}(i), A(j)) \rightarrow \text{Ext}_{A^o}^1(A/A_{\geq i}(i), A(j)), \\ 0 \rightarrow \text{Hom}_{A^o}(A_{\geq i}(i), A_{\geq j}(j)) &\rightarrow \text{Hom}_{A^o}(A_{\geq i}(i), A(j)) \rightarrow \text{Hom}_{A^o}(A_{\geq i}(i), A/A_{\geq j}(j)). \end{aligned}$$

Since A is graded Frobenius, $\text{Ext}_{A^o}^1(A/A_{\geq i}(i), A(j)) = 0$. Let $\phi \in \text{Hom}_{A^o}(A/A_{\geq i}(i), A(j))$. Since $\ell - j \geq 0$, for every $a \in A_{\ell-(j-i)}$, $a\phi(\bar{1}) = \phi(a\bar{1}) = \phi(0) = 0$. Since A is graded Frobenius and $\phi(\bar{1}) \in A(j)_{-i} = A_{j-i}$, it follows that $\phi(\bar{1}) = 0$ by Lemma 4.13, so $\phi = 0$, hence $\text{Hom}_{A^o}(A/A_{\geq i}(i), A(j)) = 0$. Since $\text{Hom}_{A^o}(A(i), A(j)) \rightarrow \text{Hom}_{A^o}(A_{\geq i}(i), A(j))$ is the restriction of the domain, the map $A_{j-i} \cong \text{Hom}_A(A(i), A(j)) \rightarrow \text{Hom}_A(A_{\geq i}(i), A(j))$ defined by $a \mapsto \cdot a$ is an isomorphism. Clearly, $\text{Hom}_{A^o}(A_{\geq i}(i), A/A_{\geq j}(j)) = 0$. Since $\text{Hom}_{A^o}(A_{\geq i}(i), A_{\geq j}(j)) \rightarrow \text{Hom}_{A^o}(A_{\geq i}(i), A(j))$ is the extension of the codomain, the map

$$\text{Hom}_{A^o}(A_{\geq i}(i), A_{\geq j}(j)) \rightarrow \text{Hom}_{A^o}(A_{\geq i}(i), A(j))$$

defined by $\cdot a \mapsto \cdot a$ is an isomorphism, hence the result. \square

If A is an AS-regular Koszul algebra over A_0 of dimension d such that $A^!$ is graded Frobenius, then there exists an isomorphism

$$\Psi : \text{End}_A \left(\bigoplus_{i=1}^d \pi\Omega^i A_0(i) \right) \rightarrow \nabla(A^!) := \begin{pmatrix} A_0^! & A_1^! & \cdots & A_{d-1}^! \\ 0 & A_0^! & \cdots & A_{d-2}^! \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0^! \end{pmatrix}$$

by Theorem 3.13, so it induces an isomorphism

$$\Psi_i : \text{End}_A(\pi\Omega^i A_0(i)) \rightarrow A_0^! \cong A_0$$

for each $1 \leq i \leq d$.

Lemma 4.15. *Let A be an AS-regular Koszul algebra over A_0 such that $A^!$ is graded Frobenius, and let $e \in A$ be an idempotent. For each $i \in \mathbb{N}^+$, the isomorphism $\Psi_i : \text{End}_A(\pi\Omega^i A_0(i)) \rightarrow A_0$ sends πe_i to e .*

Proof. For every $1 \leq i \leq d$, we have $e\Omega^i A_0(i) \cong \Omega^i(eA_0)(i) \in \text{lin } A$, so

$$\begin{array}{ccc} E_A(\Omega^i A_0(i)) & \xrightarrow{E_A(e \cdot)} & E_A(\Omega^i A_0(i)) \\ \downarrow \cong & & \downarrow \cong \\ E_A(A_0)_{\geq i}(i) & \xrightarrow{E_A(e \cdot)} & E_A(A_0)_{\geq i}(i) \end{array}$$

commutes. Moreover, since $A^!$ is graded Frobenius, we have the composition of isomorphisms

$$\begin{array}{lll} \text{End}_A(\pi\Omega^i A_0(i)) & \pi e_i = \pi e \cdot & \\ \cong \text{End}_{\text{grmod } A^!}(\overline{K}^{-1}(\Omega^i A_0(j))) & \mapsto \overline{K}^{-1}(e \cdot) & \text{(by Proposition 3.3)} \\ \cong \text{End}_{\text{grmod } A^!}(DE_A(\Omega^i A_0(i))) & \mapsto DE_A(e \cdot) & \text{(by Lemma 3.6)} \\ \cong \text{End}_{\text{grmod } A^{!o}}(E_A(\Omega^i A_0(i))) & \mapsto E_A(e \cdot) & \\ \cong \text{End}_{\text{grmod } A^{!o}}(E_A(A_0)_{\geq i}(i)) & \mapsto E_A(e \cdot) & \text{(by Lemma 3.7)} \\ \cong \text{End}_{\text{grmod } A^{!o}}(E_A(A_0)_{\geq i}(i)) & \mapsto E_A(e \cdot) & \\ \cong \text{End}_{\text{grmod } A^{!o}}(A^!_{\geq i}(i)) & \mapsto E_A(e \cdot) = \cdot(e \cdot) & \\ \cong A^!_0 & \mapsto e \cdot & \text{(by Lemma 4.14)} \\ \cong A_0, & \mapsto e. & \end{array}$$

We can check that this gives Ψ_i , so the result follows. \square

We now return to Setting 4.3.

Proposition 4.16. *In Setting 4.3, $\text{End}_{\mathcal{D}_{\text{Sg}}^{\text{gr}}(S^G)}(ve'Ue) \cong \tilde{e}'(G * \nabla(S^!))\tilde{e}'$ where*

$$\tilde{e}' = \begin{pmatrix} e' & 0 & \cdots & 0 \\ 0 & e' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e' \end{pmatrix} \in \begin{pmatrix} kG & * & \cdots & * \\ 0 & kG & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & kG \end{pmatrix} = G * \nabla(S^!).$$

Proof. By Corollary 3.15, there exists an isomorphism

$$\Psi : \text{End}_{S * G}(\pi U) \cong \nabla((S * G)^!) = \begin{pmatrix} kG & * & \cdots & * \\ 0 & kG & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & kG \end{pmatrix}.$$

By Lemma 4.15,

$$\Psi(\pi \tilde{e}') = \Psi\left(\bigoplus \pi e'_i\right) = \begin{pmatrix} \Psi_1(\pi e'_1) & 0 & \cdots & 0 \\ 0 & \Psi_2(\pi e'_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Psi_d(\pi e'_d) \end{pmatrix} = \begin{pmatrix} e' & 0 & \cdots & 0 \\ 0 & e' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e' \end{pmatrix} = \tilde{e}'.$$

Moreover one can check that \tilde{e}' is an invariant under the isomorphisms

$$\nabla((S * G)^!) \cong G * \nabla(S^!),$$

so we get

$$\begin{aligned}
 \mathrm{End}_{\mathcal{D}_{\mathrm{Sg}}^{\mathrm{gr}}(S^G)}(ve'Ue) &\cong \mathrm{End}_{S^G}(\pi e'Ue) && \text{(by Lemma 4.4)} \\
 &\cong \mathrm{End}_{S * G}(\pi e'U) && \text{(by Proposition 3.18)} \\
 &\cong \pi \bar{e}' \mathrm{End}_{S * G}(\pi U) \pi \bar{e}' && \text{(by Lemma 4.12)} \\
 &\cong \tilde{e}'(\nabla(S^! * G))\tilde{e}' \\
 &\cong \tilde{e}'(G * \nabla(S^!))\tilde{e}'.
 \end{aligned}$$

□

Theorem 4.17. *In Setting 4.3,*

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(S^G) \cong \mathcal{D}^b(\mathrm{mod} \tilde{e}'(G * \nabla(S^!))\tilde{e}')$$

as triangulated categories where

$$\tilde{e}' = \begin{pmatrix} e' & 0 & \cdots & 0 \\ 0 & e' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e' \end{pmatrix} \in \begin{pmatrix} kG & * & \cdots & * \\ 0 & kG & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & kG \end{pmatrix} = G * \nabla(S^!).$$

Proof. By Proposition 4.16,

$$\mathrm{End}_{\underline{\mathrm{CM}}^{\mathbb{Z}}(S^G)}(e'Ue) \cong \mathrm{End}_{\mathcal{D}_{\mathrm{Sg}}^{\mathrm{gr}}(S^G)}(ve'Ue) \cong \tilde{e}'(G * \nabla(S^!))\tilde{e}'.$$

Since kG is semisimple, $e'kGe'$ has finite global dimension by [10, Lemma 4.5], so

$$\tilde{e}'(G * \nabla(S^!))\tilde{e}' = \begin{pmatrix} e'kGe' & * & \cdots & * \\ 0 & e'kGe' & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e'kGe' \end{pmatrix}$$

also has finite global dimension. Since $\underline{\mathrm{CM}}^{\mathbb{Z}}(S^G)$ is an algebraic Krull-Schmidt triangulated category, the result follows from Theorem 3.9 and Theorem 4.10. □

5. EXAMPLES

The aim of this section is to provide an explicit example of Theorem 4.17. For a connected graded algebra A and $G \leq \mathrm{GrAut} A$, to check whether $A * G/(e)$ is finite dimensional over k or not, we will use a quiver presentation of $A * G/(e)$.

For the rest of this paper, k denotes an algebraically closed field of characteristic 0. Let $A = k\langle x_1, \dots, x_n \rangle / (f_1, \dots, f_h)$ be a noetherian connected graded algebra. Let G be a cyclic group generated by $g = \mathrm{diag}(\xi^{a_1}, \dots, \xi^{a_n})$ with a primitive r -th root of unity ξ and integers a_j satisfying $0 < a_j \leq r$ for any j , and $|G| = r$. We assume that G acts on A by $g \cdot x_j = \xi^{a_j} x_j$. In this case, G acts on $k\langle x_1, \dots, x_n \rangle$ by the same action.

The McKay quiver Q_G of the cyclic group G is given as follows.

- The set of vertices is $\mathbb{Z}/r\mathbb{Z}$.
- The set of arrows is $\{x_{j,i} : i - a_j \rightarrow i \mid i \in \mathbb{Z}/r\mathbb{Z}, 1 \leq j \leq n\}$.

Remark 5.1. We will often write x_j instead of $x_{j,i}$ when there is no possibility of confusion.

For any $i \in \mathbb{Z}/r\mathbb{Z}$, we define ρ_i by $\frac{1}{r} \sum_{p=0}^{r-1} \xi^{ip} g^p \in kG$. Notice that $g\rho_i = \xi^{-i}\rho_i$.

Proposition 5.2. *Define a map $\phi : k\langle x_1, \dots, x_n \rangle * G \rightarrow kQ_G$ by*

$$\phi(1 * \rho_i) = e_i \quad \text{and} \quad \phi(x_{s_1} x_{s_2} \cdots x_{s_m} * \rho_i) = x_{s_1, i - a_{s_m} \cdots - a_{s_2}} \cdots x_{s_{m-1}, i - a_{s_m}} x_{s_m, i}.$$

Then ϕ is an algebra isomorphism.

Proof. Since

$$\begin{aligned}
& (x_{s_1} \cdots x_{s_{m-1}} x_{s_m} * \rho_i)(x_{t_1} \cdots x_{t_{l-1}} x_{t_l} * \rho_{i'}) \\
&= (x_{s_1} \cdots x_{s_{m-1}} x_{s_m} * \frac{1}{r} \sum_{p=0}^{r-1} \xi^{ip} g^p)(x_{t_1} \cdots x_{t_{l-1}} x_{t_l} * \rho_{i'}) \\
&= \frac{1}{r} \sum_{p=0}^{r-1} \xi^{ip} x_{s_1} \cdots x_{s_{m-1}} x_{s_m} g^p(x_{t_1} \cdots x_{t_{l-1}} x_{t_l}) * g^p \rho_{i'} \\
&= \frac{1}{r} \sum_{p=0}^{r-1} \xi^{ip} x_{s_1} \cdots x_{s_{m-1}} x_{s_m} \xi^{p(a_{t_1} + \cdots + a_{t_l})} x_{t_1} \cdots x_{t_{l-1}} x_{t_l} * \xi^{-pi'} \rho_{i'} \\
&= \frac{1}{r} \sum_{p=0}^{r-1} \xi^{p(i-i'+a_{t_1} + \cdots + a_{t_l})} x_{s_1} \cdots x_{s_{m-1}} x_{s_m} x_{t_1} \cdots x_{t_{l-1}} x_{t_l} * \rho_{i'} \\
&= \begin{cases} x_{s_1} \cdots x_{s_{m-1}} x_{s_m} x_{t_1} \cdots x_{t_{l-1}} x_{t_l} * \rho_{i'} & \text{if } i = i' - a_{t_1} \cdots - a_{t_l} \text{ in } \mathbb{Z}/r\mathbb{Z} \\ 0 & \text{if } i \neq i' - a_{t_1} \cdots - a_{t_l} \text{ in } \mathbb{Z}/r\mathbb{Z}, \end{cases}
\end{aligned}$$

it follows that ϕ is an algebra homomorphism. It is easy to check that ϕ is bijective. \square

As an immediate consequence of the above proposition, we have the following theorem.

Theorem 5.3. *A quiver presentation of $A * G$ is given by the McKay quiver Q_G with relations*

$$\phi(f_j * \rho_i) = 0 \quad (1 \leq j \leq h, i \in \mathbb{Z}/r\mathbb{Z}).$$

The quiver presentation of $A * G$ obtained above will be denoted by the pair $(Q_{A * G}, R_{A * G})$ where $Q_{A * G}$ is the quiver Q_G , and $R_{A * G}$ is the set of relations.

Corollary 5.4. *Let $e := \frac{1}{|G|} \sum_{g \in G} g \in kG \subset A * G$ be the idempotent. Then a quiver presentation of $A * G/(e)$ is obtained from that of $A * G$ by removing the vertex 0 and all arrows incident to it.*

Proof. Since $e = \frac{1}{r} \sum_{i=0}^{r-1} g^i = \rho_0$, this is clear. \square

The quiver presentation of $A * G/(e)$ obtained above will be denoted by the pair $(Q_{A * G/(e)}, R_{A * G/(e)})$ where $Q_{A * G/(e)}$ is the quiver obtained by removing the vertex 0 from Q_G , and $R_{A * G/(e)}$ is the set of relations.

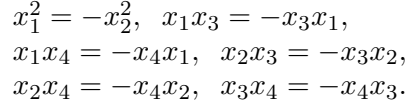
Using Theorem 5.3 and Corollary 5.4, we now present a nontrivial example. We consider the case when the acting group G is a subgroup of $\text{HSL}(S)$ but not a subgroup of $\text{SL}(d, k)$.

Example 5.5. Let S be $k\langle x_1, x_2, x_3, x_4 \rangle$ having six defining relations

$$x_1^2 + x_2^2, \quad x_1 x_3 + x_3 x_1, \quad x_1 x_4 + x_4 x_1, \quad x_2 x_3 + x_3 x_2, \quad x_2 x_4 + x_4 x_2, \quad x_3 x_4 + x_4 x_3,$$

with $\deg x_1 = \deg x_2 = \deg x_3 = \deg x_4 = 1$. Then S is a noetherian AS-regular Koszul algebra over k of dimension 4. Let G be a cyclic group generated by $g = \text{diag}(1, -1, -1, -1)$. Then g defines a graded algebra automorphism of S , so G naturally acts on S . Clearly $|G| = 2$. Moreover one can check that the homological determinant of g is equal to 1 (although $\det g \neq 1$), so it follows that S^G is AS-Gorenstein of dimension 4.

By Theorem 5.3, the quiver presentation $(Q_{S * G}, R_{S * G})$ is given by


$$\begin{array}{c} 1 \\ \curvearrowright \\ x_1 \end{array} \quad x_1^2 = 0,$$

The Koszul dual $S^!$ is $k\langle x_1, x_2, x_3, x_4 \rangle$ having ten defining relations

with $\deg x_1 = \deg x_2 = \deg x_3 = \deg x_4 = 1$. Then a quiver presentation of the Beilinson algebra $\nabla(S^!)$ is given as follows.

$$\begin{aligned} x_1^2 &= x_2^2, \quad x_3x_1 = x_1x_3, \quad x_4x_1 = x_1x_4, \\ x_3x_2 &= x_2x_3, \quad x_4x_2 = x_2x_4, \quad x_4x_3 = x_3x_4, \\ x_2x_1 &= x_1x_2 = x_3^2 = x_4^2 = 0. \end{aligned}$$

$$\begin{array}{ccc}
 (0,0) \xrightarrow{x_1} (1,0) \xrightarrow{x_1} (2,0) \xrightarrow{x_1} (3,0) & & x_1^2 = x_2^2, \ x_3x_1 = x_1x_3, \\
 \begin{array}{c} \nearrow x_2 \\ \nearrow x_3 \\ \searrow x_4 \\ \searrow x_4 \end{array} & & x_1x_4 = x_4x_1, \ x_3x_2 = x_2x_3, \\
 & & x_2x_4 = x_4x_2, \ x_4x_3 = x_3x_4, \\
 (0,1) \xrightarrow{x_1} (1,1) \xrightarrow{x_1} (2,1) \xrightarrow{x_1} (3,1) & & x_2x_1 = x_1x_2 = x_3^2 = x_4^2 = 0.
 \end{array}$$
$$\begin{array}{c}
\begin{array}{ccccccc}
& \xrightarrow{x_2 x_3} & & \xrightarrow{x_2 x_4} & & \xrightarrow{x_3 x_4} & \\
(0, 1) & \xrightarrow{-x_1} & (1, 1) & \xrightarrow{-x_1} & (2, 1) & \xrightarrow{-x_1} & (3, 1) \\
& \xleftarrow{x_2 x_3} & & \xleftarrow{x_2 x_4} & & \xleftarrow{x_3 x_4} &
\end{array} \\
\end{array}
\quad
\begin{array}{l}
x_2 x_3 x_1 = x_1 x_2 x_3 = 0 \\
x_2 x_4 x_1 = x_1 x_2 x_4 = 0 \\
x_3 x_4 x_1 = x_1 x_3 x_4 \\
x_1^3 = 0
\end{array}
\tag{5.1}$$
$$\underline{\mathbf{CM}}^{\mathbb{Z}}(S^G) \cong \mathcal{D}^b(\text{mod } kQ/(R))$$

by Theorem 4.17.

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